Faster algorithms for the characteristic polynomial

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- Improves previous complexity by a $\log n$ factor,
- Optimal reduction to Matrix multiplication.
- Practical efficiency. E.g. over $\mathbb{Z}_{547\,909}$:

n	500	5000	15 000
`LinBox`	0.91s	4m44s	2h20m
`magma-2.13`	1.27s	15m32s	7h28m

Outline

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Pre-Strassen age

Leverrier 1840: trace of powers of A, and Newton's formula

- improved/rediscovered by Souriau, Faddeev, Frame and Csanky
- $\mathcal{O}\left(n^4\right)$, based on Matrix multiplication
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Hessenberg 1942: transformation to quasi-upper triangular and determinant expansion formula.
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Keller-Gehrig 1985, alg.2: inspired by Danilevskii, block operations

- $\mathcal{O}\left(n^{\omega}\right)$
- but only valid with generic matrices

Outline

Definition (degree d Krylov matrix of one vector v)

$$K = \begin{bmatrix} v & Av & \dots & A^{d-1}v \end{bmatrix}$$

Property

$$A \times K = K \times \underbrace{\begin{bmatrix} 0 & & & * \\ 1 & & & * \\ & \ddots & & * \\ & & 1 & * \end{bmatrix}}_{C_{P_{min}^{A,v}}}$$

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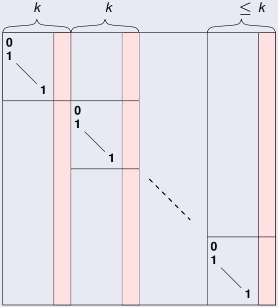
[Keller-Gehrig, alg. 2] : $K^{-1}AK$ in $\mathcal{O}(n^\omega)$ for A generic

Definition (degree k Krylov matrix of several vectors v_i)

$$K = \left[\begin{array}{ccc|ccc|c|ccc} v_1 & \dots & A^{k-1}v_1 & v_2 & \dots & A^{k-1}v_2 & \dots & v_l & \dots & A^{k-1}v_l \end{array}\right]$$

Property

$A \times K = K \times$ 

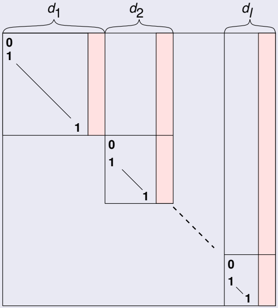
Fact (Shift Hessenberg form)

If $(d_1, \ldots d_l)$ is lexicographically maximal such that

$$K = \left[\begin{array}{ccc|c|ccc} v_1 & \ldots & A^{d_1-1}v_1 & \ldots & v_l & \ldots & A^{d_l-1}v_l \end{array}\right]$$

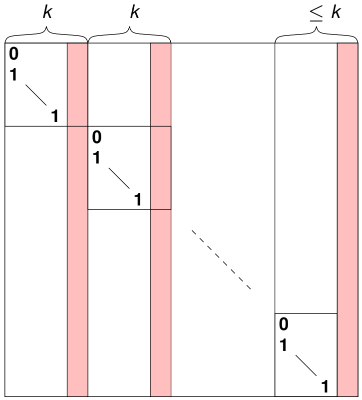
is non-singular, then

$A \times K = K \times$



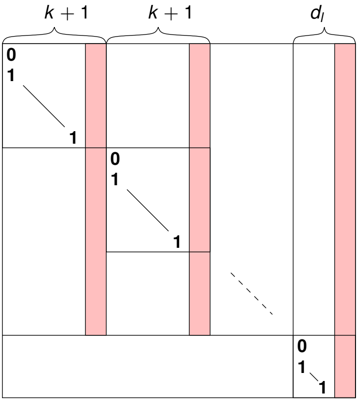
Principle

k-shifted form:



Principle

$k+1$-shifted form:



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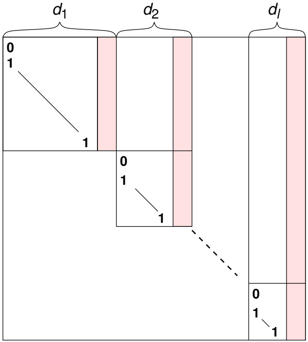
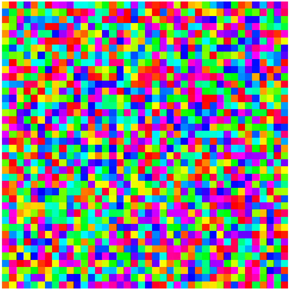
- each completed block appears in the increasing degree order
- until the shifted Hessenberg form is obtained:



Example



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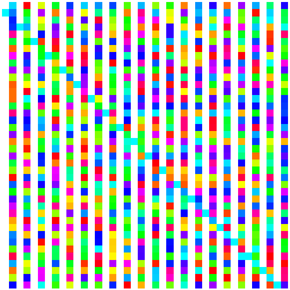

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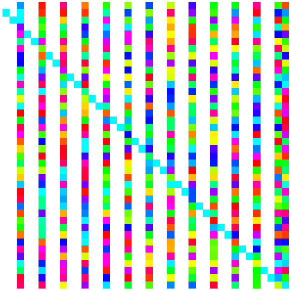

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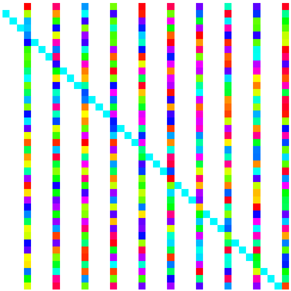

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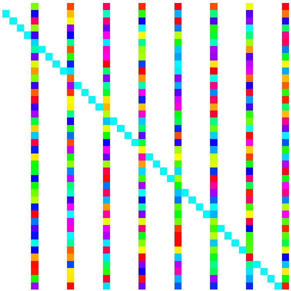

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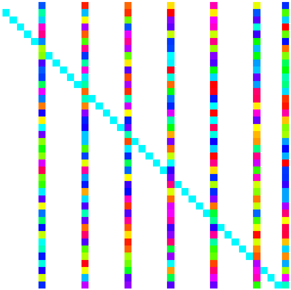

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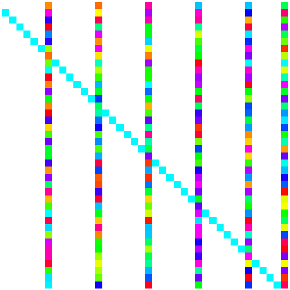

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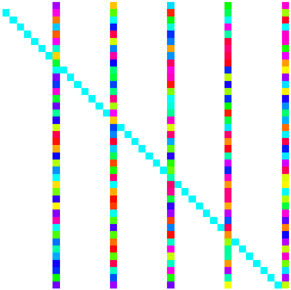

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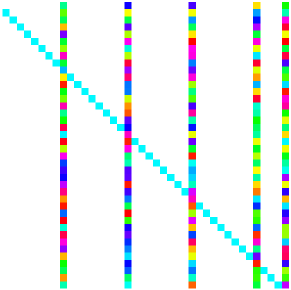

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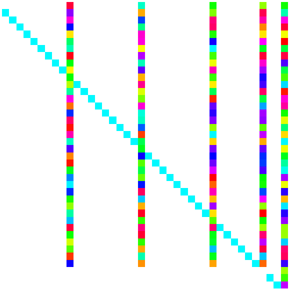

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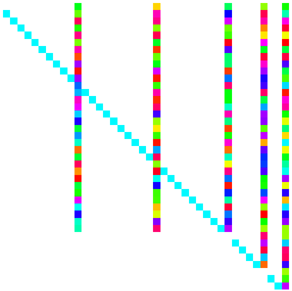

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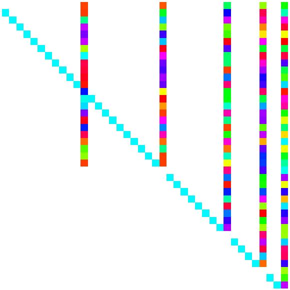

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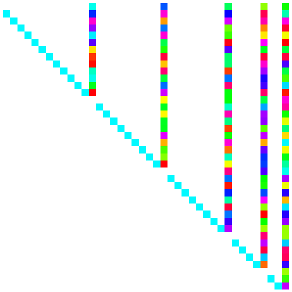

Lemma

If $\#F > 2n^2$, the transformation will succeed with high probability. Failure is detected.

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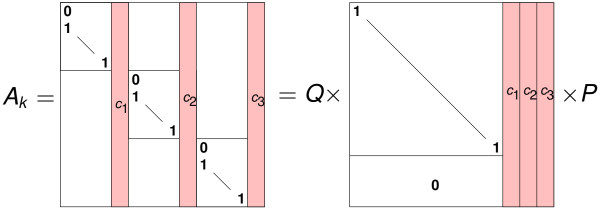
How to use fast matrix arithmetic ?

Permutations: compressing the dense columns



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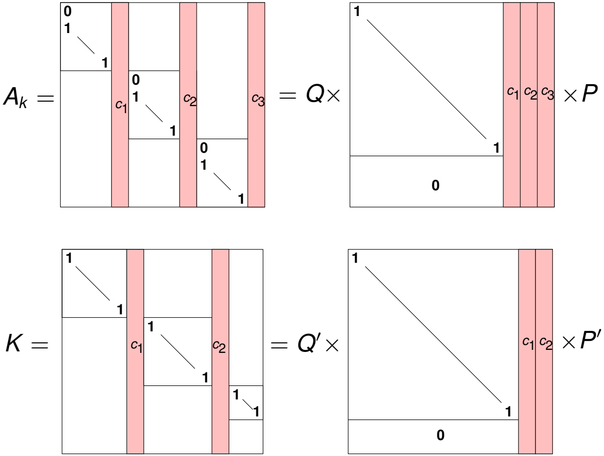

Reduction to Matrix multiplication

Similarity transformation:

$$K^{-1}AK = Q'^T \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} P'^T Q \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} PQ' \begin{bmatrix} I & * \\ 0 & * \end{bmatrix} P'$$

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$$K^{-1}AK = Q'^{T}\left(\begin{bmatrix} I & * \\ 0 & * \end{bmatrix}\left(P'^{T}Q\left(\begin{bmatrix} I & * \\ 0 & * \end{bmatrix}\left(PQ'\begin{bmatrix} I & * \\ 0 & * \end{bmatrix}\right)\right)\right)\right)P'$$

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$$\sum_{k=1}^{n} k\left(\frac{n}{k}\right)^{\omega} = n^{\omega}\sum_{k=1}^{n}\frac{1}{k^{\omega-1}} = \mathcal{O}\left(n^{\omega}\right)$$

A new type of reduction

$xI_n - A$

dimension $= n$
degree $= 1$

$\downarrow$

dimension $= 1$
degree $= n$

$det(xI_n - A)$

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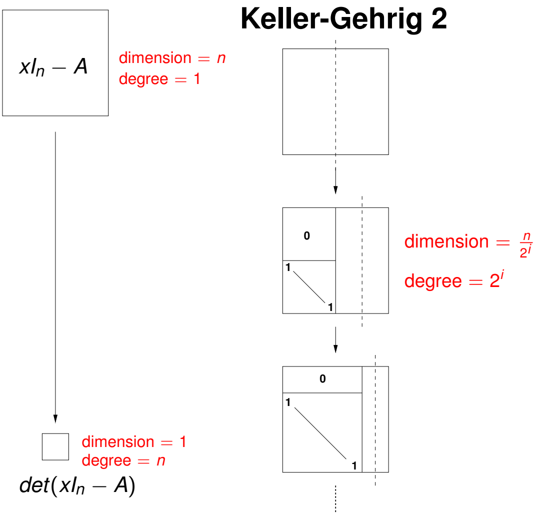

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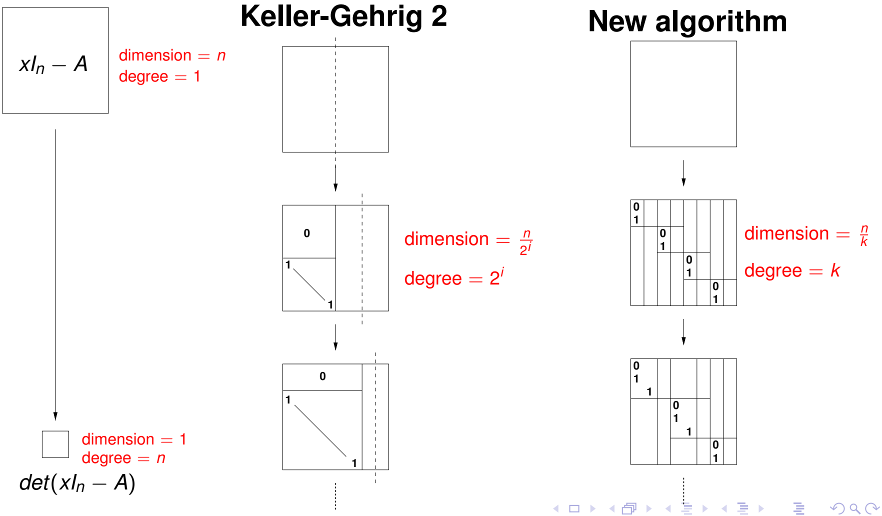

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Improving the preconditioning

The preconditioning phase:

$$A \leftarrow U^{-1}AU$$

for a random matrix U.

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$$V = \begin{bmatrix} W & AW & \dots & A^{c-1}W \end{bmatrix}$$

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Property

$V^{-1}AV$ is in c shifted form.

Efficiency balancing parameter

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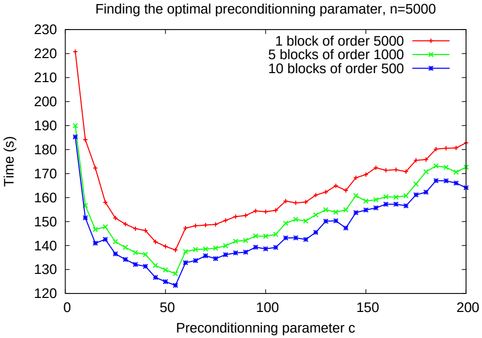
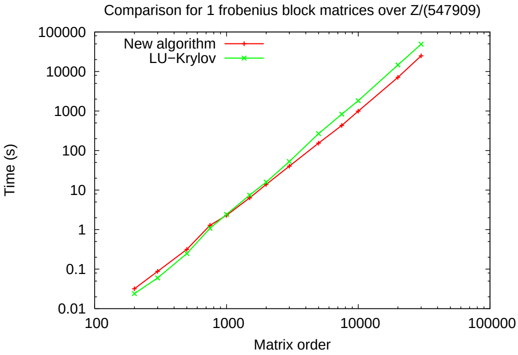


Experiments

n	LU-Krylov	New algorithm
200	0.024	0.032
300	0.06s	0.088s
500	0.248s	0.316s
750	1.084s	1.288s
1000	2.42s	2.296s
5000	267.6s	153.9s
10 000	1827s	991s
20 000	14 652s	7097s
30 000	48 887s	24 928s

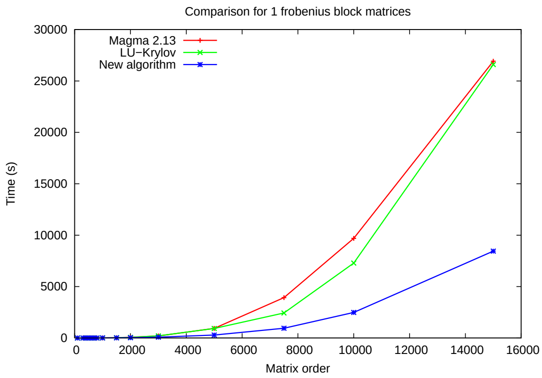
Computation time for 1 Frobenius block matrices, Itanium2-64 1.3Ghz, 192Gb

Experiments



Timing comparison between the new algorithm and LU-Krylov, logarithmic

Comparison to Magma and previous LinBox



Conclusion and perspectives

Results:

- Las Vegas reduction to matrix multiplication,
- The Frobenius normal form is easily derivable in $\mathcal{O}(n^\omega)$...
- ...but no transformation matrix
- Adaptive combination with block Krylov in practice.

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Still to be done:

- Condition on the size of the field is a limitation. Eberly's algorithm ?
- Ideally: derandomization ? (deterministic)
- Unification with matrix polynomial algorithms