High Performance and Reliable Algebraic Computing

Soutenance d'habilitation à diriger des recherches

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Université Joseph Fourier (Grenoble 1)

November 25, 2014

Rapporteurs :	Daniel Augot, Mark Giesbrecht, Laura Grigori,	Examinateurs :	Jean-Guillaume Dumas, Jean-Charles Faugère, Erich L. Kaltofen, Brigitte Plateau.

Introduction

Computer Algebra

Computing exactly over $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, \mathsf{GF}(q), \mathsf{K}[X]$.

- Symbolic manipulations.
- Applications where all digits matter:

- breaking Discrete Log Pb. in quasi-polynomial time [Barbulescu & *al.* 14],
- building modular form databases to test the BSD conjecture [Stein 12],
- formal verification of Hales' proof of Kepler conjecture [Hales 05].

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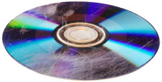
Efficiency mostly rely on linear algebra over $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$.

Introduction

Coding theory



Protecting information against alteration:

- deep space communication,
- data storage,
- fault tolerance of large scale computations.

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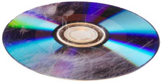

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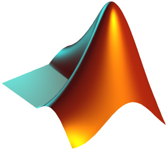
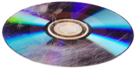
Numerical linear algebra



Computing fast with approximations:

- delivering flops to most scientific computations for over 60 years,
- `LinPack`: benchmark for the top 500 supercomputers,
- impacts nowadays computer architectures.

Interactions



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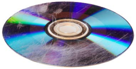

[Berlekamp 68, Massey 69]

Krylov methods

[Wiedemann 86]: sparse linear system solving over $\mathbb{F}_q$

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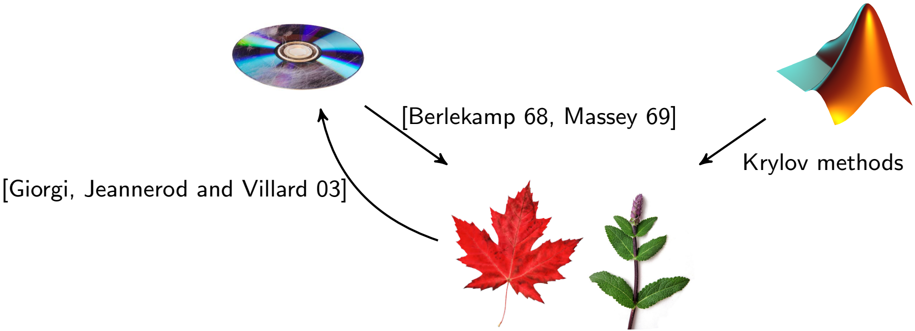

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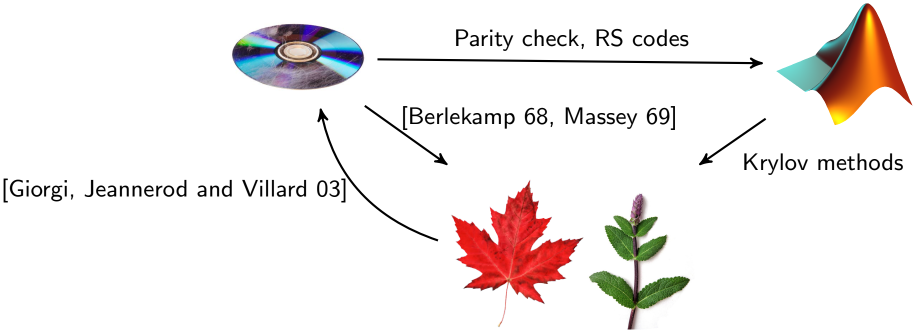

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[Huang and Abraham 84]: Algorithm Based Fault Tolerance (ABFT)

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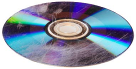
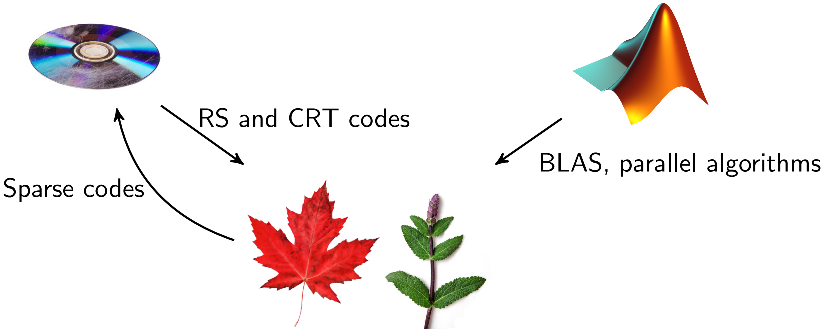


BLAS, parallel algorithms

Contributions:

- design of high performance linear algebra kernels,

Interactions



Contributions:

- design of high performance linear algebra kernels,
- fault tolerant computer algebra.

Outline

1. Design of High Performance Exact Linear Algebra Kernels
   - Matrix multiplication
   - Gaussian elimination
   - Rank profiles
   - Characteristic polynomial
2. Coding Theory for Fault Tolerant Computer Algebra
   - Approximation problems
   - Dense polynomial evaluation codes
   - Rational function codes
   - Sparse evaluation codes

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Matrix Product

[Strassen 69]: $O(n^{2.807})$

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Other operations

[Strassen 69]: Inverse in $O(n^\omega)$

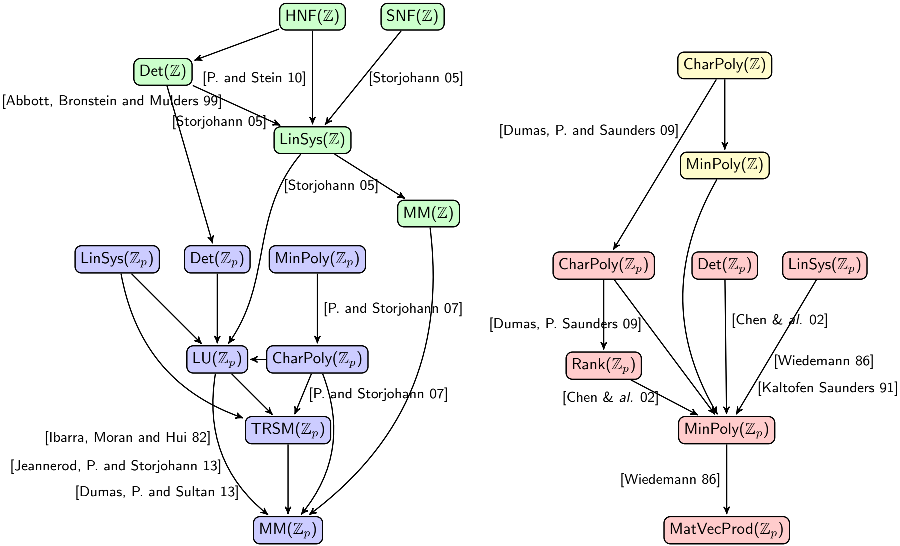
[Schönhage 72]: QR in $O(n^\omega)$

[Bunch, Hopcroft 74]: LU in $O(n^\omega)$

[Ibarra & *al.* 82]: Rank in $O(n^\omega)$

[Keller-Gehrig 85]: CharPoly in $O(n^\omega \log n)$

Reductions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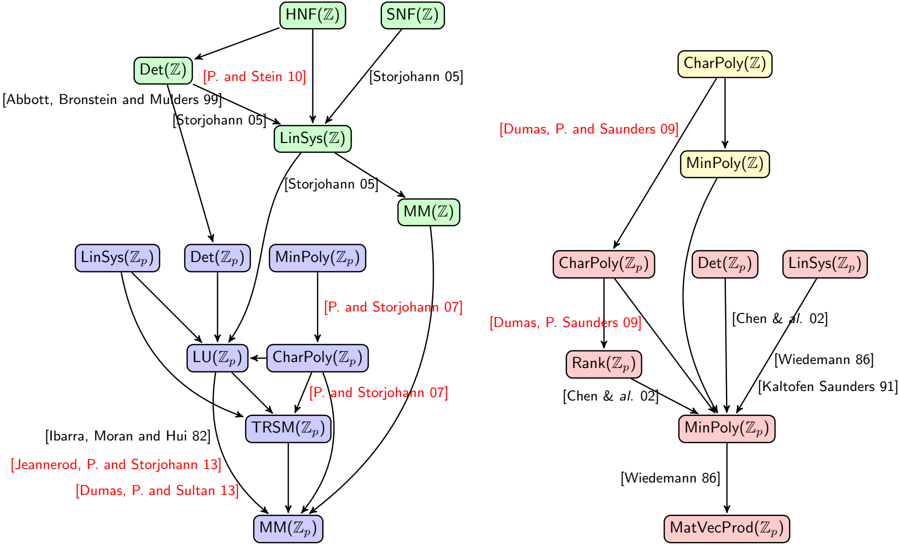
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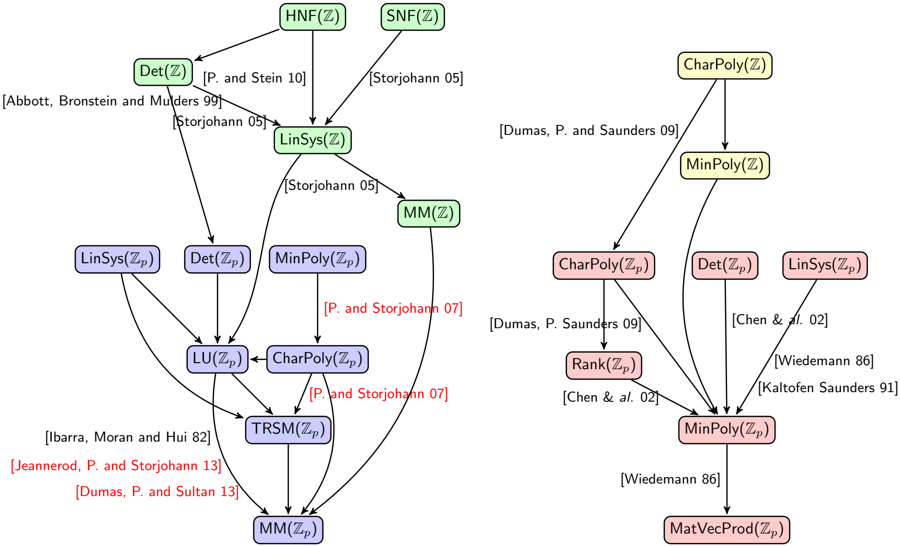
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- ✓ building blocks **in theory** happen to be the most efficient routines **in practice**

$\rightsquigarrow$ reduction trees are still relevant

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Roadmap

1. Tune building blocks (MatMul)
2. Improve existing reductions (LU, Echelon)
   - ▹ leading constants
   - ▹ memory footprint
3. Produce new reduction schemes (CharPoly, Rank Profiles)

Design of parallel exact linear algebra

ANR HPAC project:

1. efficient kernels for exact linear algebra on SMP
2. DSL, runtime as a plugin and composition
3. attacking large scale challenges from cryptography

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- cost invariant wrt. splitting
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  $\rightsquigarrow$ fine grain

  $\rightsquigarrow$ block iterative algorithms
- regular task load
- Numerical stability constraints

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[Broquedis, Danjean and Gautier 12]: `libkomp` based on `XKaapi`

Matrix Multiplication over $\mathbb{Z}/p\mathbb{Z}$

Ingedients [Dumas, Gautier and P. 02]

- Compute over $\mathbb{Z}$ and delay modular reductions

$$\rightsquigarrow \quad k\left(\frac{p-1}{2}\right)^2 < 2^{\text{mantissa}}$$

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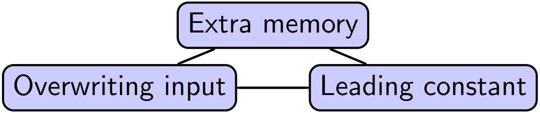
with memory efficient schedules [Boyer, Dumas, P. and Zhou 09]

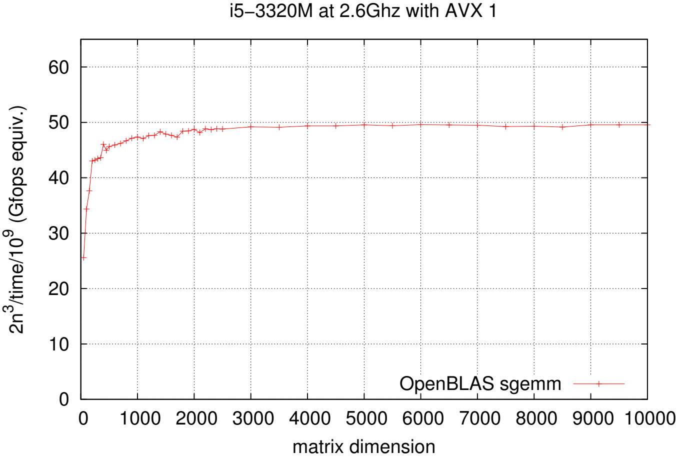
Tradeoffs:



Fully in-place in

$7.2n^{2.807} + \dots$

Sequential Matrix Multiplication



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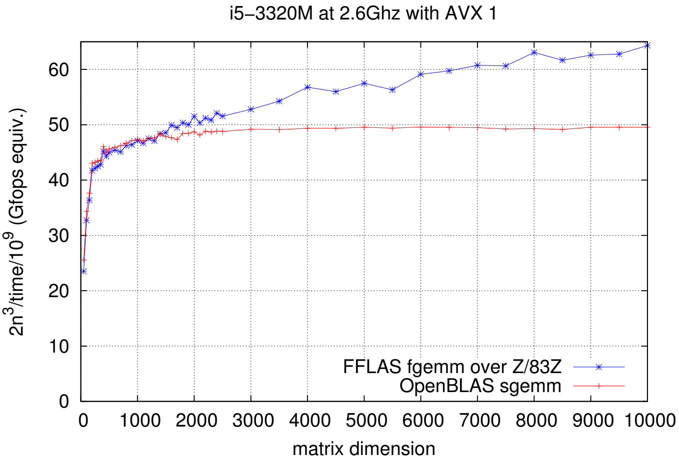

$p = 83$, $\rightsquigarrow$ 1 mod / 10000 mul.

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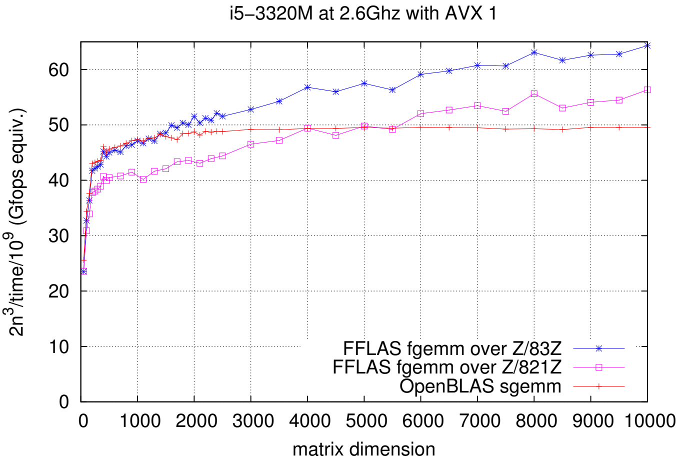

$p = 83$, $\rightsquigarrow$ 1 mod / 10000 mul.

$p = 821$, $\rightsquigarrow$ 1 mod / 100 mul.

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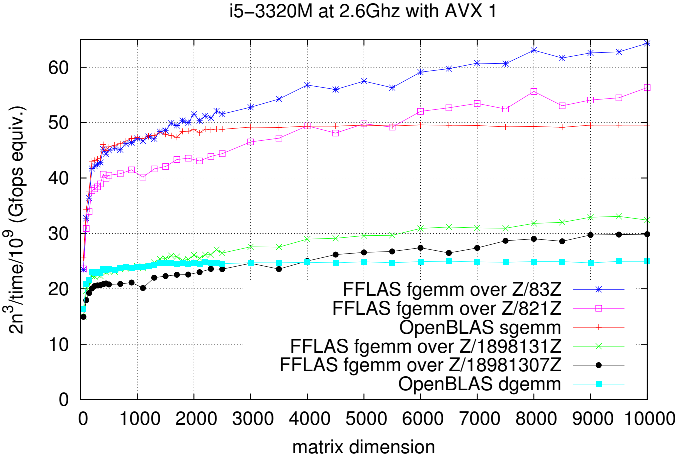

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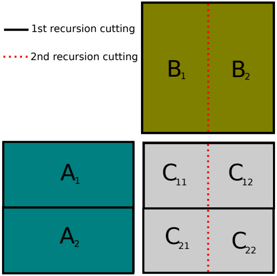
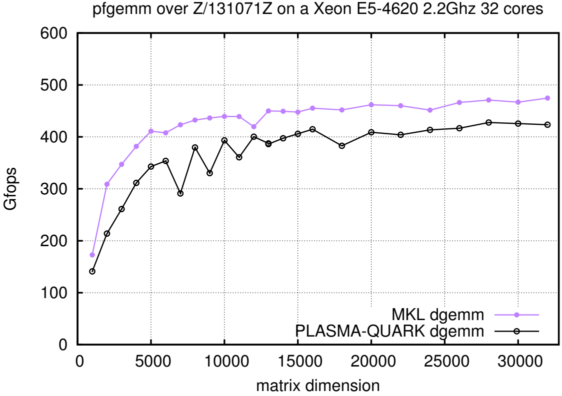
$p = 18981307$, $\rightsquigarrow$ 1 mod / 100 mul.

Parallel matrix multiplication

Dumas, Gautier, P. and Sultan 14





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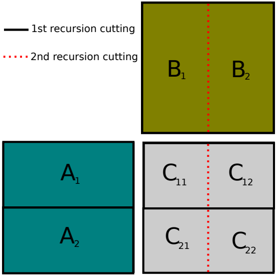

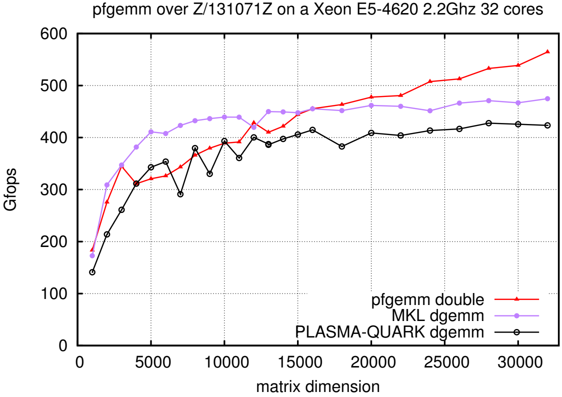

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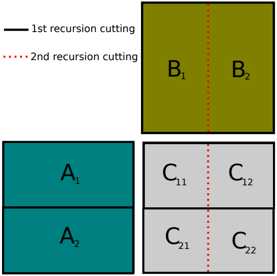
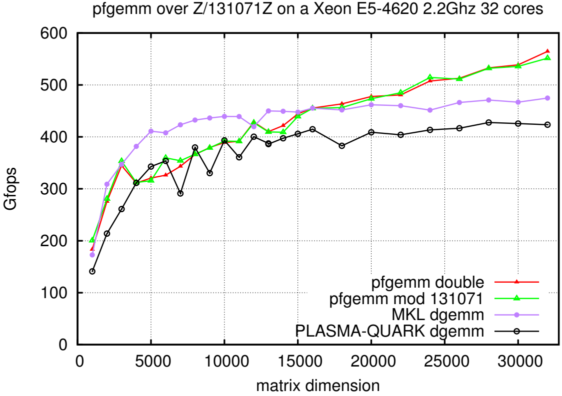
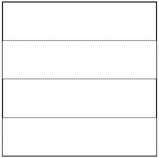
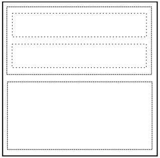




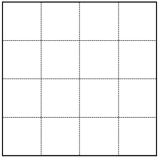
Gaussian elimination



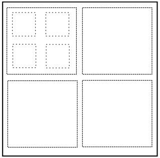
Slab iterative
`LAPACK`



Slab recursive
`FFLAS-FFPACK`



Tile iterative
`PLASMA`



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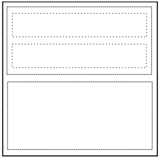

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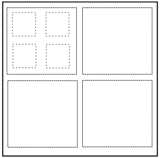

Tile recursive
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- Prefer recursive algorithms

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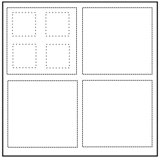

Tile recursive
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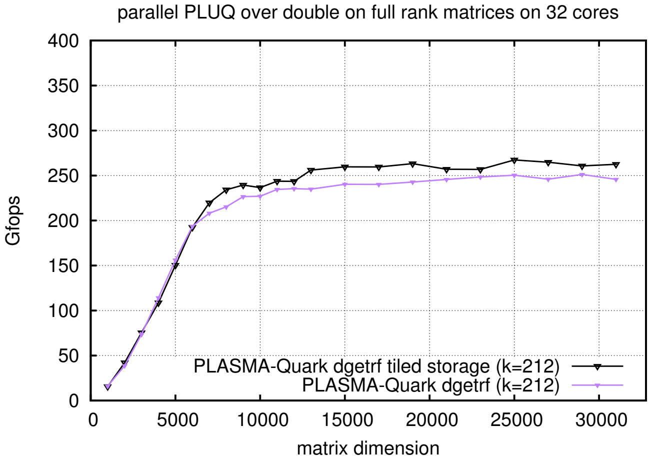
- Prefer recursive algorithms
- Better data locality

Full rank Gaussian elimination

Dumas, Gautier, P. and Sultan 14

Comparing numerical efficiency (no modulo)



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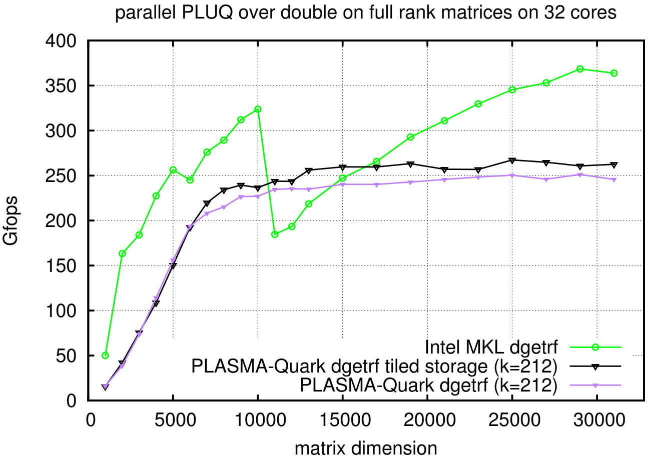

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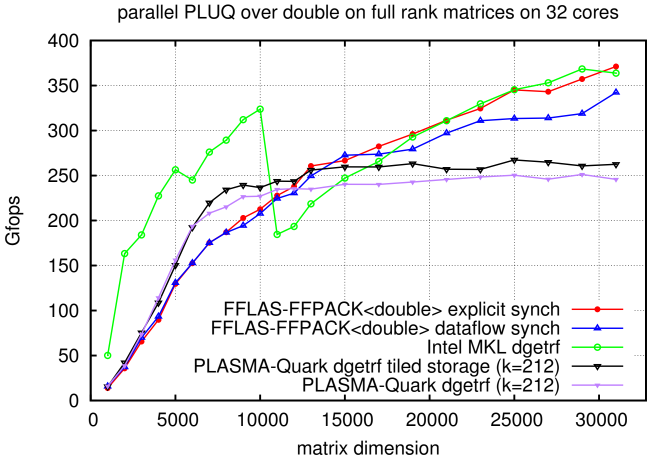

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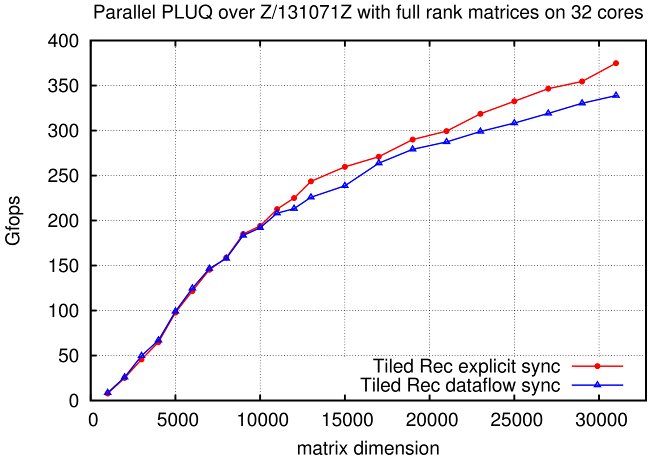
Over the finite field $\mathbb{Z}/131071\mathbb{Z}$



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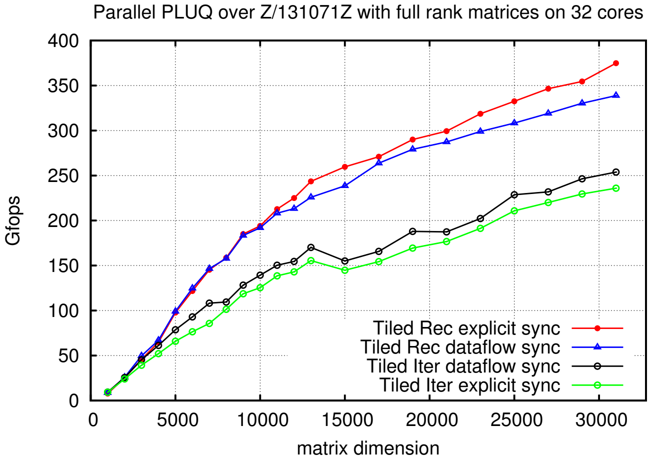
Dumas, Gautier, P. and Sultan 14

Over the finite field $\mathbb{Z}/131071\mathbb{Z}$



Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m \times n}, r = \operatorname{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal sub-sequence of $(1, \ldots, m)$ of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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RowRP = {1,2,4}

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RowRP = {1,2,4}
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- Major invariant of a matrix (echelon form)
- Gröbner basis computations (Macaulay matrix) [Faugère 99, 02]
- Krylov methods

Computing rank profiles

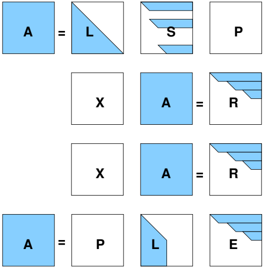
Via Gaussian elimination revealing echelon forms:

[Ibarra, Moran and Hui 82]

[Keller-Gehrig 85]

[Storjohann 00]

[Jeannerod, P. and Storjohann 13]



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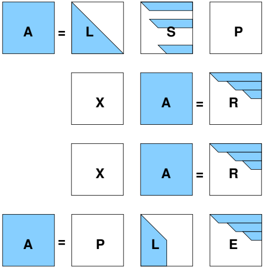


Lessons learned (or what we thought was necessary):

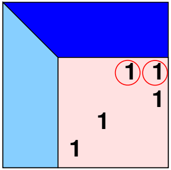
- treat rows in order
- exhaust all columns before considering the next row
- **slab** block splitting required (recursive or iterative)
  $\rightsquigarrow$ similar to partial pivoting

Pivoting strategies revealing rank profiles

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row



Search	Row perm.	Col. perm.	RowRP	ColRP	Tiles possible
Row order					

Pivoting strategies revealing rank profiles

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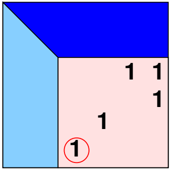

Search	Row perm.	Col. perm.	RowRP	ColRP	Tiles possible
Row order Col. order					

Pivoting strategies revealing rank profiles

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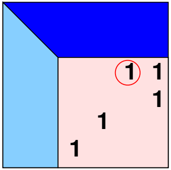
Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex order: first non-zero on the first non-zero row



Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	Tiles possible
Row order Col. order					
Lexico.					

Pivoting strategies revealing rank profiles

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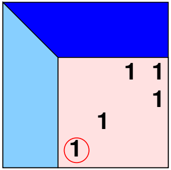
Lex/RevLex order: first non-zero on the first non-zero row/col



Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	Tiles possible
Row order Col. order					
Lexico.					
Rev. lex.					

Pivoting strategies revealing rank profiles

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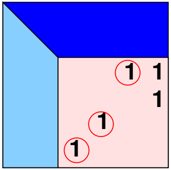
Product order: first non-zero in the (i, j) leading sub-matrix



Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	Tiles possible
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Lexico.					
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Permutation

- Transpositions

Search	Row perm.	Col. perm.	RowRP	ColRP	Tiles possible
Row order	Transposition	Transposition	✓		✗
Col. order	Transposition	Transposition		✓	✗
Lexico.	Transposition	Transposition	✓		✗
Rev. lex.	Transposition	Transposition		✓	✗
Product					

Pivoting strategies revealing rank profiles

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Permutation

- Transpositions
- Cylcic Rotations

Search	Row perm.	Col. perm.	RowRP	ColRP	Tiles possible
Row order	Transposition	Transposition	✓		✗
Col. order	Transposition	Transposition		✓	✗
Lexico.	Transposition	Transposition	✓		✗
Rev. lex.	Transposition	Transposition		✓	✗
Product	Rotation	Transposition	✓		✓
Product	Transposition	Rotation		✓	✓

Pivoting strategies revealing rank profiles

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Row order	Transposition	Transposition	✓			✗
Col. order	Transposition	Transposition		✓		✗
Lexico.	Transposition	Transposition	✓			✗
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Product	Rotation	Transposition	✓			✓
Product	Transposition	Rotation		✓		✓
Product	Rotation	Rotation	✓	✓	✓	✓

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Lexico.	Transposition	Transposition	✓			✗
Lexico.	Transposition	Rotation	✓	✓	✓	✗
Rev. lex.	Transposition	Transposition		✓		✗
Rev. lex.	Rotation	Transposition	✓	✓	✓	✗
Product	Rotation	Transposition	✓			✓
Product	Transposition	Rotation		✓		✓
Product	Rotation	Rotation	✓	✓	✓	✓

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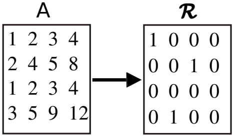
Rev. lex.	Transposition	Transposition		✓		✗
Rev. lex.	Rotation	Transposition	✓	✓	✓	✗
Product	Rotation	Transposition	✓			✓
Product	Transposition	Rotation		✓		✓
Product	Rotation	Rotation	✓	✓	✓	✓

Computing all rank profiles at once

Dumas, P. and Sultan 13

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



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Theorem

- *RowRP and ColRP read directly on $\mathcal{R}(A)$*
- *Same holds for any (i,j)-leading submatrix.*

A

1	2	3	4
2	4	5	8
1	2	3	4
3	5	9	12

→

$\mathcal{R}$

1	0	0	0
0	0	1	0
0	0	0	0
0	1	0	0

RowRP $= \{1\}$
ColRP $= \{1\}$

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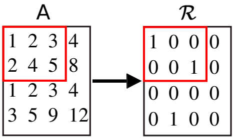

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ColRP = {1,3}

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RowRP = {1,4}
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$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \quad \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \quad \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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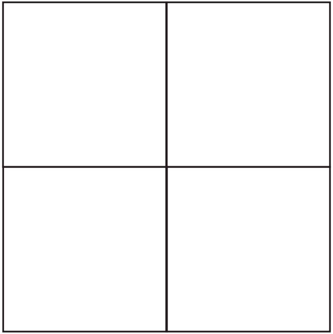
With appropriate pivoting: $\Pi_{P,Q} = \mathcal{R}(A)$

A tiled recursive algorithm

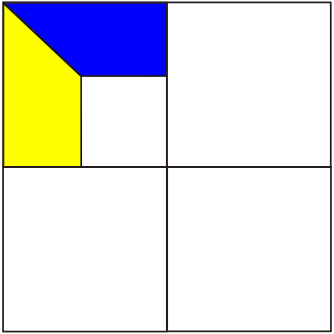
Dumas, P. and Sultan 13



2×2 block splitting

A tiled recursive algorithm

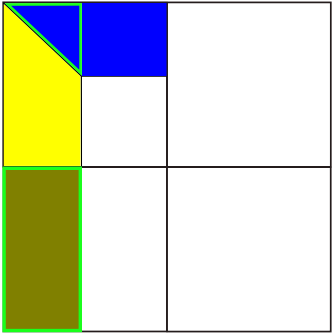
Dumas, P. and Sultan 13



Recursive call

A tiled recursive algorithm

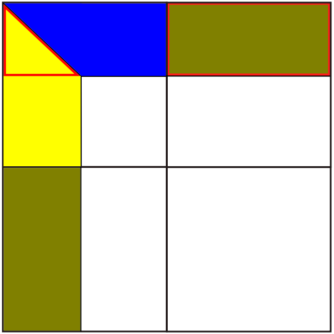
Dumas, P. and Sultan 13



TRSM: $B \leftarrow BU^{-1}$

A tiled recursive algorithm

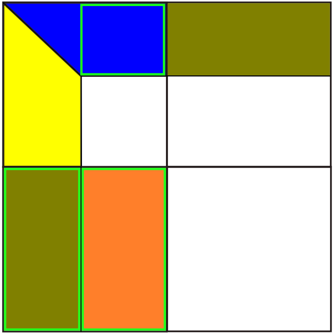
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

A tiled recursive algorithm

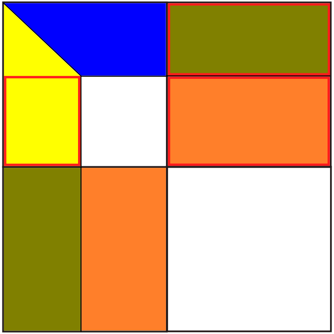
Dumas, P. and Sultan 13



`MatMul`: $C \leftarrow C - A \times B$

A tiled recursive algorithm

Dumas, P. and Sultan 13



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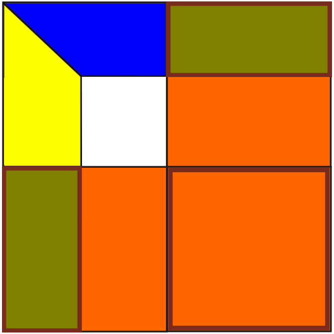
A tiled recursive algorithm

Dumas, P. and Sultan 13



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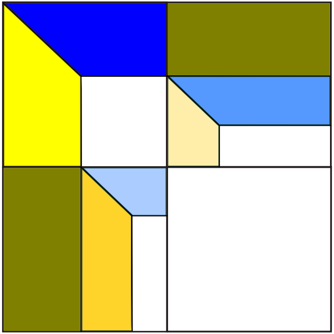
A tiled recursive algorithm

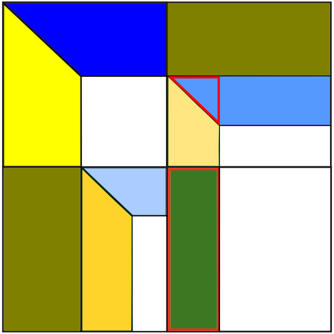
Dumas, P. and Sultan 13



2 independent recursive calls

A tiled recursive algorithm

Dumas, P. and Sultan 13



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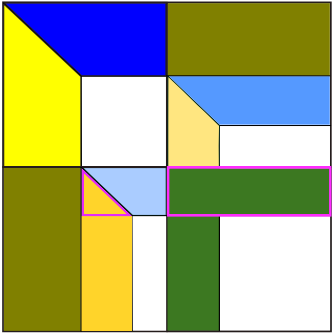
A tiled recursive algorithm

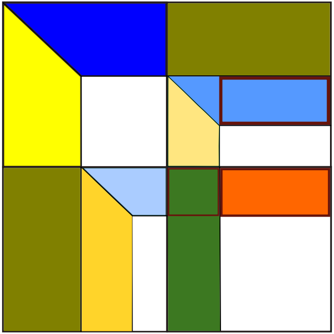
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

A tiled recursive algorithm

Dumas, P. and Sultan 13



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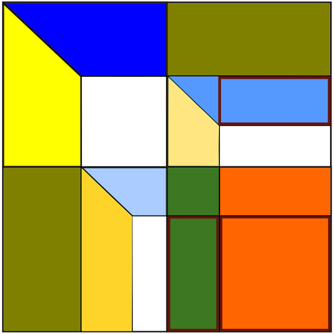
A tiled recursive algorithm

Dumas, P. and Sultan 13



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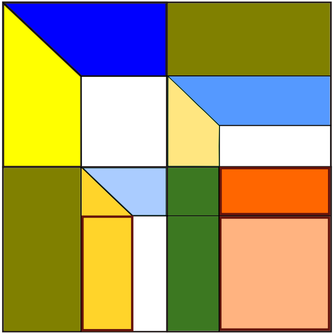
A tiled recursive algorithm

Dumas, P. and Sultan 13



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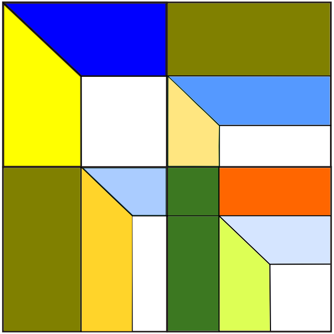
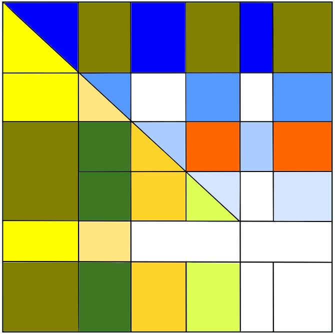
Dumas, P. and Sultan 13



Recursive call

A tiled recursive algorithm

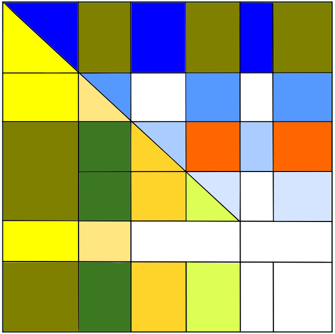
Dumas, P. and Sultan 13



Puzzle game (block cyclic rotations)

A tiled recursive algorithm

Dumas, P. and Sultan 13



- $O(mnr^{\omega-2})$ (degenerating to $2/3n^3$)
- computing col. and row rank profiles of all leading sub-matrices
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Computing the characteristic polynomial

Motivation

- Connection with the Frobenius normal form
- Krylov methods at large
- Graph invariants
- Crucial step in modular form computations

The last missing reduction

[Danilevskiĭ 37], [Hessenberg 42]	CharPoly, deterministic	$O(n^3)$
[Keller-Gehrig 85]	CharPoly, deterministic	$O(n^\omega \log n)$
[Giesbrecht 93]	Frobenius form, Las-Vegas probabilistic	$O(n^\omega \log n)$
[Augot, Camion 94]	Frobenius form, deterministic	$O(n^3 \#\text{inv factors})$
[Storjohann 00]	Frobenius form, deterministic	$O(n^3)$ or $O(n^\omega \log n \log\log n)$

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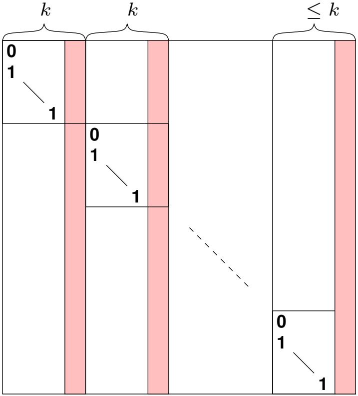
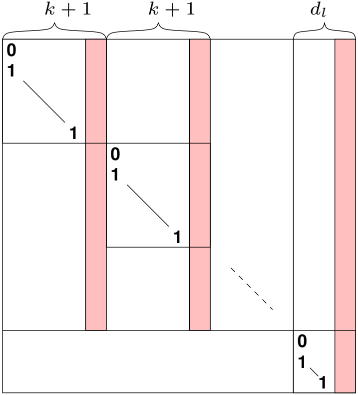
k-shifted form:



P. and Storjohann ISSAC'07

$k+1$-shifted form:



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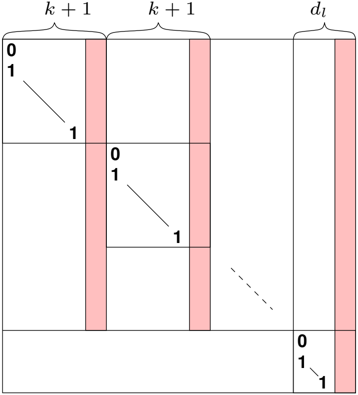
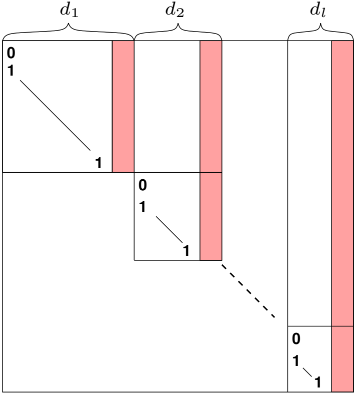


- From k to $k+1$-shifted in $O(n(\frac{n}{k})^{\omega-1})$
- Compute iteratively from a 1-shifted form
- Invariant factors appear by increasing degree

P. and Storjohann ISSAC'07

Hessenberg polycyclic:



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P. and Storjohann ISSAC'07

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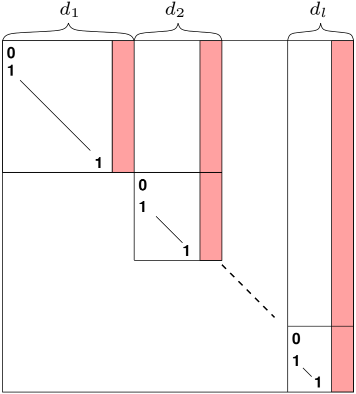

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$$n^\omega \sum_{k=1}^{n} \left(\frac{1}{k}\right)^{\omega-1} \leq \zeta(\omega-1)n^\omega = O(n^\omega)$$

P. and Storjohann ISSAC'07

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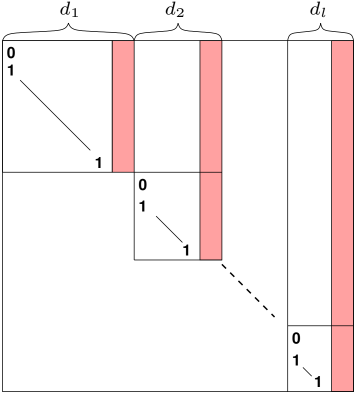

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P. and Storjohann ISSAC'07

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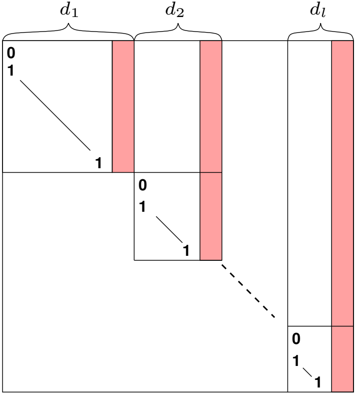

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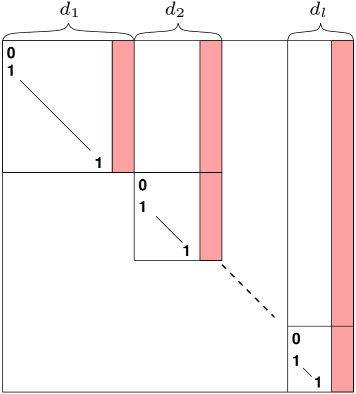
n	1000	2000	5000	10000
magma-v2.19-9	1.38s	24.28s	332.7s	2497s
fflas-ffpack	0.532s	2.936s	32.71s	219.2s

P. and Storjohann ISSAC'07

Hessenberg polycyclic:



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$\times 7.5$

$\times 6.7$

Outline

Fault Tolerance

Reliability of large scale distributed computing

	Peak	Mean Time To Error	Mean Time To Failure
Blue Waters	14 Pflops	15min	$\approx 1/2$ day
Tsubame 2	2.3 Pflops	?	15.8h

- Disk crash, hardware/software failures $\rightsquigarrow$ hard errors
- Bitflip in main or cache memory $\rightsquigarrow$ soft/silent errors

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Trust in outsourced computations (P2P, Cloud, Volunteer, etc)

Byzantine error model:

- a corrupted node is not always wrong
- black-listing is not an option

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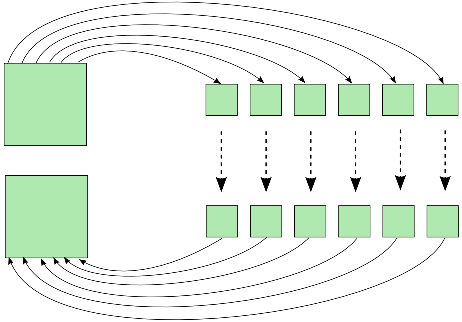
Algorithm Based Fault Tolerance:

exploit the algebraic specificity of the algorithm to embed redundancy.

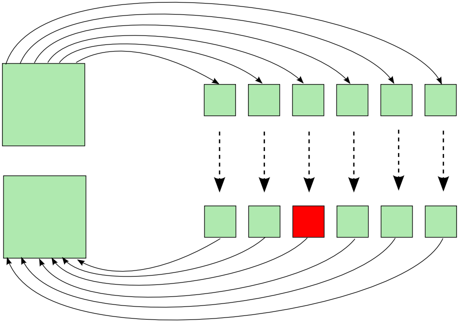
ABFT using error correcting codes

Computations $\Leftrightarrow$ Communication



ABFT using error correcting codes

Unsecure Computations ⇔ Noisy Communication



⇝ Choice of the parallelization algorithm determines

- the communication channel
- the error model

Evaluation-interpolation schemes

Polynomial evaluation

$$\begin{array}{rccc} \mathsf{Ev}_{(x_0,\dots,x_{n-1})}: & \mathsf{K}_{<n}[X] & \longrightarrow & \mathsf{K}^n \\ & f & \longmapsto & (f(x_0),\dots,f(x_{n-1})) \end{array}$$

for $x_0,\dots,x_{n-1}$ distinct.

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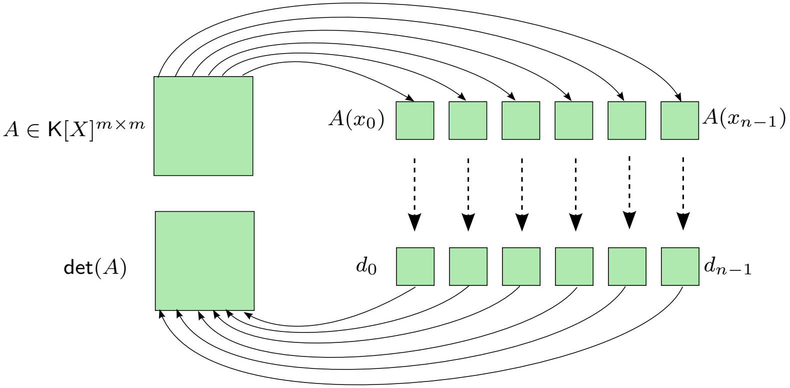
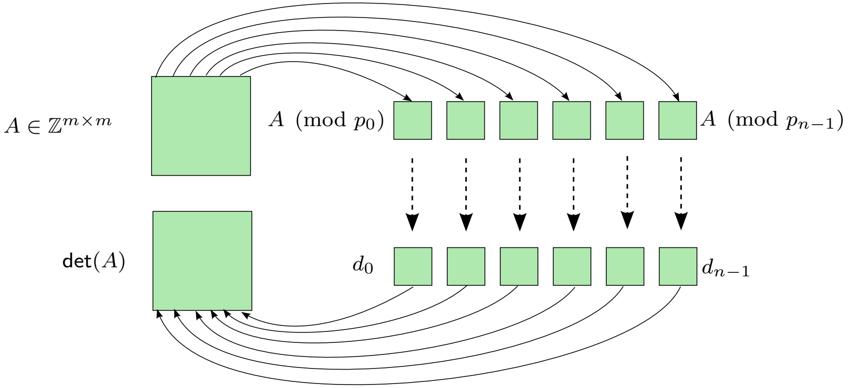


Evaluation-interpolation schemes

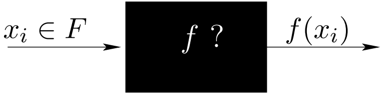
Chinese Remainder Theorem

$$\begin{array}{rccc} \mathrm{Ev}_{(p_0,\dots,p_{n-1})}: & \mathbb{Z}_{<p_0\times\dots\times p_{n-1}} & \longrightarrow & \mathbb{Z}_{p_0}\times\dots\times\mathbb{Z}_{p_{n-1}} \\ & m & \longmapsto & (m \bmod p_0, \dots, m \bmod p_{n-1}) \end{array}$$

for $p_1, \dots, p_n$ pairwise co-prime.



Making evaluation-interpolation schemes fault tolerant



Problem

Recover an unknown function f, given as a black-box, from its evaluations.

Making evaluation-interpolation schemes fault tolerant

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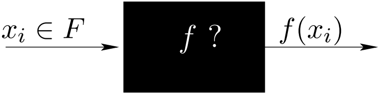
Additional knowledge on the model

Dense polynomial: degree bound

Making evaluation-interpolation schemes fault tolerant



$f = \sum_{i=1}^{t} c_i X^{d_i}$

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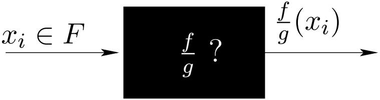
Making evaluation-interpolation schemes fault tolerant



$$f = \sum_{i=0}^{d_f} f_i X^i, \; g = \sum_{i=0}^{d_g} g_i X^i$$

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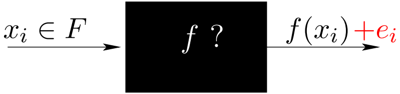
Making evaluation-interpolation schemes fault tolerant



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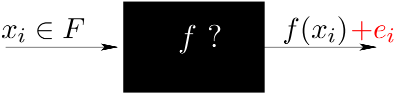
Trust in the evaluations

- errors (outliers)
- approximations: numerical noise

Making evaluation-interpolation schemes fault tolerant



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Rational function reconstruction

Problem (RFR: Rational Function Reconstruction)

Given $A, B \in \mathsf{K}[X]$ *with* $\deg B < \deg A = n$,
Find $f \in \mathsf{K}_{\leq d_f}[X]$, $g \in \mathsf{K}_{\leq n-d_f-1}[X]$ *such that*

$$f = gB \mod A.$$

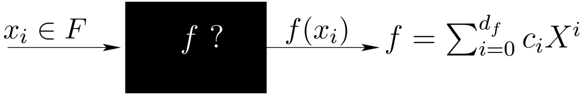
Fact

The Extended Euclidean Algo. run on (A, B) *and terminated when* $\deg f_j \leq d_f < \deg f_{j-1}$, *produces* $f_j = u_j A + v_j B$ *s.t.*

1. (f_j, v_j) *is a solution to the RFR problem.*
2. *it is minimal: any other solution* (f, g) *is of the form*

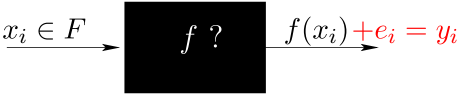
$$f = qf_j, \quad g = qv_j \quad \text{for } q \in \mathsf{K}[X].$$

Dense polynomial interpolation



without error: polynomial interpolation (Lagrange, Newton, etc).

Dense polynomial interpolation with errors



$$f = \sum_{i=0}^{d_f} c_i X^i$$

without error: polynomial interpolation (Lagrange, Newton, etc).

with errors: Reed-Solomon decoding

- Erroneous interpolant: $h = \mathsf{Interp}((y_i, x_i))$
- Error locator polynomial: $\Lambda = \prod_{e_i \neq 0} (X - x_i)$

$$\Lambda f = \Lambda h \mod \prod_{i=0}^{n-1} (X - x_i)$$

Dense polynomial interpolation with errors

$x_i \in F$ → **f ?** → $f(x_i) + e_i = y_i$ $f = \sum_{i=0}^{d_f} c_i X^i$

without error: polynomial interpolation (Lagrange, Newton, etc).

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- Erroneous interpolant: $h = \text{Interp}((y_i, x_i))$
- Error locator polynomial: $\Lambda = \prod_{e_i \neq 0}(X - x_i)$

$$\underbrace{\Lambda f}_{N} = \underbrace{\Lambda}_{D} h \mod \prod_{i=0}^{n-1}(X - x_i)$$

Rational Reconstruction Problem:

$(\Lambda f, \Lambda)$ is a minimal solution ⇝ computed by Ext. Euclidean Algorithm

$$f = f_j / v_j.$$

Correction capacity

Unique decoding of t errors whenever: $n \geq \deg f + 2E + 1$

Bounding the degree

- $\deg f$ rarely known *a priori*; bound $d_f \geq \deg f$ often pessimistic
- Early termination:

  without errors: add evaluations until interpolant stabilizes

  with errors: no stabilization

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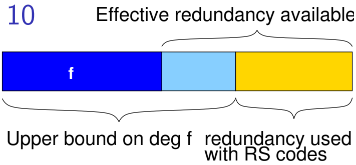
Parameter oblivious decoding

Khonji, P., Roch, Roche and Stalinsky 10

$\rightsquigarrow$ how to use all available redundancy?



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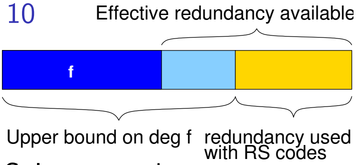
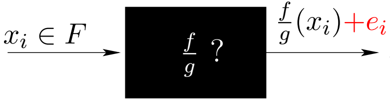


⇝ how to use all available redundancy?

⇝ list decoder exploring all length n Reed-Solomon codes

Dense rational function interpolation with errors



$f = \sum_{i=0}^{d_f} f_i X^i, \ g = \sum_{i=0}^{d_g} g_i X^i$

$$\underbrace{\Lambda f}_{N} = \underbrace{\bar{\Lambda}\bar{g}}_{D}\, h \mod \prod_{y_i \neq \infty} (X - x_i)$$

Rational Reconstruction Problem

$(\Lambda f, \bar{\Lambda}\bar{g})$ is a minimal solution $\rightsquigarrow$ computed by Ext. Euclidean Algorithm.

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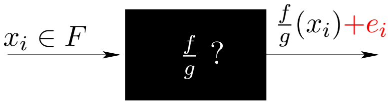

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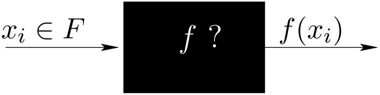
Correction capacity

Unique decoding of E errors whenever $n \geq d_f + d_g + 2E + 1$

- smoothly supports evaluations at poles (even erroneous ones)
- parameter oblivious decoding applies

Sparse interpolation

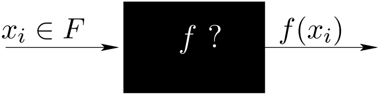


$f = \sum_{i=1}^{t} c_i X^{d_i}$

Without error: [Prony 1795] [Ben-Or and Tiwari 88]

- sample in a geometric progression: $y_i = f(\alpha^i)$

Sparse interpolation

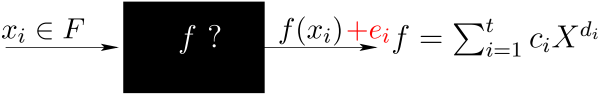


$$f = \sum_{i=1}^{t} c_i X^{d_i}$$

Without error: [Prony 1795] [Ben-Or and Tiwari 88]

- sample in a geometric progression: $y_i = f(\alpha^i)$
- [Blahut'84]: the seq. $(y_0, y_1, ...)$ has linear complexity t
- and is generated by $\Lambda(X) = \prod_{i=1}^{t}(X - \alpha^{d_i})$
- Berlekamp-Massey algo. on $2t$ terms $\rightsquigarrow d_i$
- Vandermonde system $\rightsquigarrow c_i$

Sparse interpolation with errors



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- Vandermonde system $\rightsquigarrow c_i$

With errors: rule of thumb:

- find a clean sub-sequence of $2t$ terms free of error

Unique decoding by majority rule Berlekamp-Massey

Comer, Kaltofen and P. 12

Necessary condition for unique decoding: $n \geq 2t(2E+1)$

(a_i)	$\Lambda(z)$	$f(z)$
$x = (\overbrace{\overline{0}}^{(t-1) zeros}, 1, \overline{0}, 1, \ldots, \overline{0}, 1)$	$z^t - 1$	$\frac{1}{t}\sum_{i=0}^{t-1} z^{2i\frac{m}{2t}}$

Unique decoding by majority rule Berlekamp-Massey

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$y = (\underbrace{\bar{0}}_{(t-1)zeros}, 1, \bar{0}, -1, \ldots, \bar{0}, -1)$	$z^t + 1$	$\frac{-1}{t}\sum_{i=0}^{t-1} z^{(2i+1)\frac{m}{2t}}$

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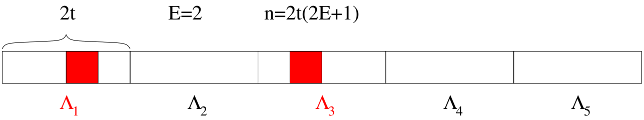
Sufficient condition for unique decoding: $n \leq 2t(2E+1)$



Unique decoding by majority rule Berlekamp-Massey

Comer, Kaltofen and P. 12

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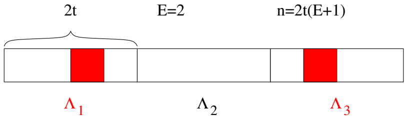
Sufficient condition for list decoding: $n \leq 2t(E+1)$



List decoding: using affine sub-sequences

Kaltofen and P. 14

$f(\alpha^0)$	$f(\alpha^1)$	$f(\alpha^2)$	$f(\alpha^3)$	$f(\alpha^4)$	$f(\alpha^5)$	$f(\alpha^6)$	$f(\alpha^7)$	$f(\alpha^8)$
$f(\alpha^0)$		$f(\alpha^2)$		$f(\alpha^4)$		$f(\alpha^6)$		$f(\alpha^8)$
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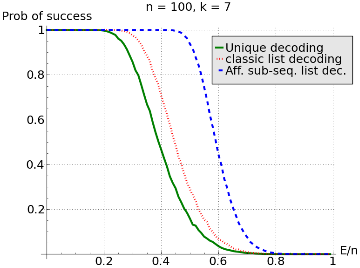

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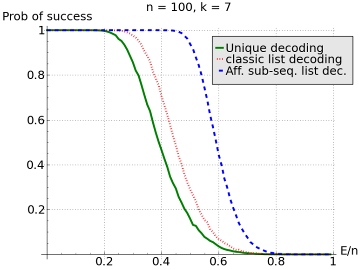

Difficult worst case analysis

- [Erdös and Turan 36]: size of the largest subset of $\{1 \ldots n\}$ not containing k terms in arithmetic progression
- [Szeremedi 75]: arithmetic prog. are dense

$$n - \frac{n}{(\log\log n)^{1/2^{2^{k+9}}}} \leq E \leq \frac{n}{k-2} \log_k \frac{n}{k-2}.$$

Towards better decoding capacities

Unique decoding: $n \geq 2t(2E+1)$

List decoding (basic): $n \geq 2t(E+1)$

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Improved conditions for unique decoding

Descartes' rule of signs: Over $\mathsf{K} = \mathbb{R}_{>0}$: $n \geq 2t + 2E + 1$

Irreducibility of cyclotomic polynomials:

- Over $\mathsf{K} = \mathbb{C}$: $n \geq 2t\frac{\log \deg f}{\log 2t} + 2E + 1$
- Over $\mathbb{F}_q^{(p_1)} \times \cdots \times \mathbb{F}_q^{(p_n)}$: $n \geq 2t\frac{\log \deg f}{\log 2t} + 2E + 1$

- No known decoding algorithm
- Makes the list decoding algo. a unique decoding one

Conclusion

Design framework for high performance exact linear algebra

Asymptotic reduction > algorithm tuning > building block implementation

- Favor **tiled recursive** algorithms
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  $\rightsquigarrow$ $O(r^\omega + (m+n+|A|)^{1+o(n)})$? [Storjohann and Yang 14]

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- **Recursive tasks** and **coarse grain** parallelization
  $\rightsquigarrow$ Light weight task workstealing management required (`libkomp`)
  $\rightsquigarrow$ Need for an improved recursive **dataflow** scheduling

Conclusion

Fault tolerance based on evaluation codes

- RS and CRT codes extended to rational fractions
  $\rightsquigarrow$ smooth generalization
- Parameter oblivious decoding for early termination schemes
  $\rightsquigarrow$ parameter oblivious **list-decoding**? [Wu 08]
- Sparse evaluation codes
  $\rightsquigarrow$ Gap between best correction radius and existing algorithm

Perspectives

Large scale distributed exact linear algebra

- reducing communications [Demmel, Grigori and Xiang 11]
- sparse and hybrid (Boyer and Vialla) [Faugère and Lachartre 10]
- combine genericity and efficiency to attack crypto. challenges

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Thank you