Computing the Rank Profile Matrix

Clément Pernet
joint work with Jean-Guillame Dumas and Ziad Sultan

Laboratoire de l'Informatique du Parallélisme,
Univ. Grenoble Alpes, Univ. de Lyon, CNRS, Inria.

AriC Seminar – October 8, 2015

Gaussian elimination in computer algebra

Swiss army knife for applications:

Matrix factorization **(LU decomposition)**

- Solving linear systems
- Computing determinants

Gaussian elimination in computer algebra

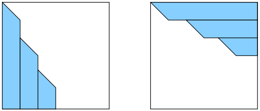
Swiss army knife for applications:

Matrix factorization **(LU decomposition)**

- Solving linear systems
- Computing determinants

Computing linear dependencies **(Echelon structure)**

- Basis of vector spaces (Krylov iteration)
- Echelon structure of the Macaulay matrix (Gröbner basis)



Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathrm{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal list of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank profiles

Definition (Row Rank Profile: RowRP)

Given $A \in \mathsf{K}^{m \times n}, r = \mathsf{rank}(A)$.

informally: *first* r linearly independent rows

formally: lexico-minimal list of r indices of linearly independant rows.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathrm{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3}

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \mathrm{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3} → Generic ColRP.

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \text{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3} $\rightarrow$ Generic ColRP.

Generic Rank Profile: first r leading principal minors $\neq$ 0

Generic rank profile $\not\Leftarrow\!\!\Rightarrow$ Generic Row RP and Genric ColRP

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

RowRP = ColRP = {1,2}

Rank profiles

Definition (Column Rank Profile: ColRP)

Given $A \in \mathsf{K}^{m\times n}, r = \text{rank}(A)$.

informally: *first* r linearly independent columns

formally: lexico-minimal list of r linearly independant columns.

Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank = 3
RowRP = {1,2,4}
ColRP = {1,2,3} $\rightarrow$ Generic ColRP.

Generic Rank Profile: first r leading principal minors $\neq 0$

Generic rank profile $\not\Leftarrow\!\Rightarrow$ Generic Row RP and Genric ColRP

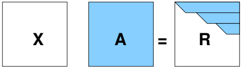
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

RowRP = ColRP = {1,2}
But $|A_{1,1}| = 0$

Relation to echelon forms

Transformation to echelon form

$\forall A\ \exists X$ non-singular s.t.

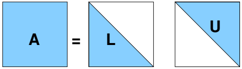


Relation to echelon forms:

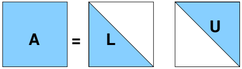
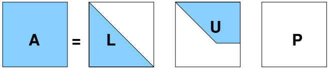
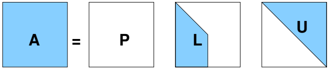
- ColRP unchanged by left multiplication with an invertible matrix

ColRP $=$ pivot columns in the **row** echelon form

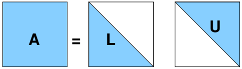
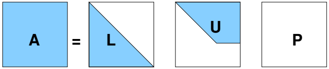
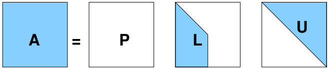
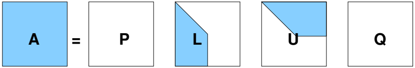
Triangular Matrix decompositions and rank profiles

Decomposition	Exists for	Unique
	Generic rank profile	Y

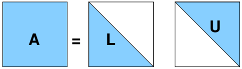
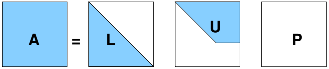
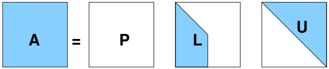
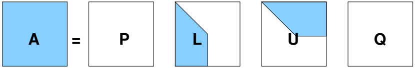
Triangular Matrix decompositions and rank profiles

Decomposition	Exists for	Unique
	Generic rank profile	Y
	Generic row rank profile	N
	Generic col rank profile	N

Triangular Matrix decompositions and rank profiles

Decomposition	**Exists for**	**Unique**
	Generic rank profile	Y
	Generic row rank profile	N
	Generic col rank profile	N
	Any matrix	N

Triangular Matrix decompositions and rank profiles

Decomposition	Exists for	Unique
	Generic rank profile	Y
	Generic row rank profile	N
	Generic col rank profile	N
	Any matrix	N

$\rightarrow P, Q$ may reveal row and/or col rank profiles.

Computing rank profiles

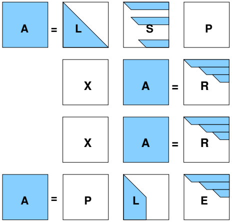
Via Gaussian elimination revealing row echelon forms:

[Ibarra, Moran and Hui 82]

[Keller-Gehrig 85]

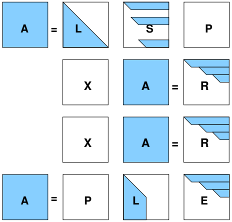
[Storjohann 00]

[Jeannerod, P. and Storjohann 13]



Computing rank profiles

Via Gaussian elimination revealing row echelon forms:



[Ibarra, Moran and Hui 82]

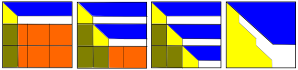
[Keller-Gehrig 85]

[Storjohann 00]

[Jeannerod, P. and Storjohann 13]

Lessons learned (or what we thought was necessary):

- treat rows in order
- exhaust all columns before considering the next row
- **slab** block splitting (recursive or iterative)



$\rightsquigarrow$ similar to partial pivoting

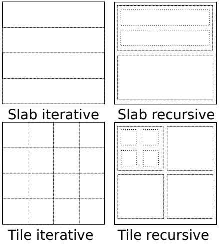
Motivation

Need more flexible blocking

Slab blocking

- leads to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead ?



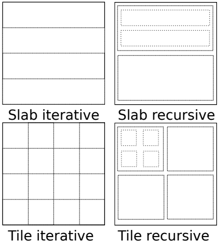
Motivation

Need more flexible blocking

Slab blocking

- leads to inefficient memory access patterns
- is harder to parallelize

Tile blocking instead ?



Gathering linear independence invariants

Two ways to look at a matrix (looking left or right):

- Row rank profile, column echelon form
- Column rank profile, row echelon form

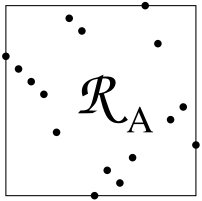
Unique invariant?

Outline

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m\times n}$.
There exists a unique, $m \times n$, *rank*(A)*-sub-permutation matrix* $\mathcal{R}^A$ *of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of* A.



The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m \times n}$.
There exists a unique, $m \times n$, *rank*(A)*-sub-permutation matrix* $\mathcal{R}^A$ *of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of* A.

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m\times n}$.
There exists a unique, $m \times n$*, rank*(A)*-sub-permutation matrix* $\mathcal{R}^A$ *of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of* A*.*

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m \times n}$.
There exists a unique, $m \times n$, *rank*(A)*-sub-permutation matrix* $\mathcal{R}^A$ *of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of* A.

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rank profile Matrix

Theorem

Let $A \in \mathrm{F}^{m\times n}$.
There exists a unique, $m \times n$, rank(A)-sub-permutation matrix $\mathcal{R}^A$ of which every leading sub-matrix has the same rank as the corresponding leading sub-matrix of A.

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of the rank profile matrix

Properties

- *A invertible $\Rightarrow \mathcal{R}^A$ is a permutation matrix*
- *A is square with generic rank profile $\Rightarrow \mathcal{R}^A = I_n$*
- *RowRP(A) = RowSupport$(\mathcal{R}^A)$*
- *ColRP(A) = ColSupport$(\mathcal{R}^A)$*

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{matrix} \text{RowRP} = \{1,3,4\} \\ \text{ColRP} = \{1,2,4\} \end{matrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Properties of the rank profile matrix

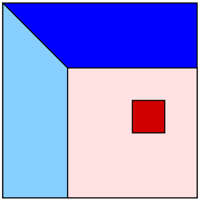
Properties

- *A invertible $\Rightarrow \mathcal{R}^A$ is a permutation matrix*
- *A is square with generic rank profile $\Rightarrow \mathcal{R}^A = I_n$*
- *RowRP(A) = RowSupport$(\mathcal{R}^A)$*
- *ColRP(A) = ColSupport$(\mathcal{R}^A)$*
- *RowRP$(A_{1..i,1..j})$ = RowSupport$({\mathcal{R}^A}_{1..i,1..j})$*
- *ColRP$(A_{1..i,1..j})$ = ColSupport$({\mathcal{R}^A}_{1..i,1..j})$*

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 4 & 7 \end{bmatrix} \qquad \begin{matrix} \text{RowRP} = \{1,3\} \\ \text{ColRP} = \{1,2\} \end{matrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outline

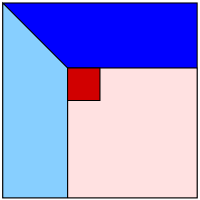
Anatomy of a PLUQ decomposition



Four types of elementary operations:

Search: finding a pivot

Anatomy of a PLUQ decomposition

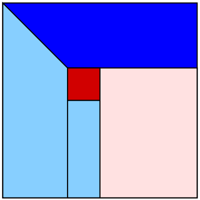


Four types of elementary operations:

Search: finding a pivot

Permutation: moving the pivot to the main diagonal

Anatomy of a PLUQ decomposition



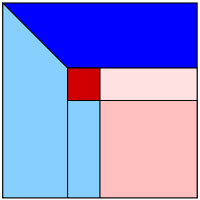
Four types of elementary operations:

Search: finding a pivot

Permutation: moving the pivot to the main diagonal

Normalization: computing L: $l_{i,k} \leftarrow \frac{a_{i,k}}{a_{k,k}}$

Anatomy of a PLUQ decomposition



Four types of elementary operations:

Search: finding a pivot

Permutation: moving the pivot to the main diagonal

Normalization: computing L: $l_{i,k} \leftarrow \frac{a_{i,k}}{a_{k,k}}$

Update: applying the elimination $a_{i,j} \leftarrow a_{i,j} - \frac{a_{i,k}a_{k,j}}{a_{k,k}}$

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

Update: no impact on the decomposition, only in the scheduling:

- iterative, tile/slab iterative, recursive,
- left/right looking, Crout

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

Update: no impact on the decomposition, only in the scheduling:

- iterative, tile/slab iterative, recursive,
- left/right looking, Crout

Search: defines the first r values of P and Q

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

Update: no impact on the decomposition, only in the scheduling:

- iterative, tile/slab iterative, recursive,
- left/right looking, Crout

Search: defines the first r values of P and Q

Permutation: impacts all values of P and Q

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

Update: no impact on the decomposition, only in the scheduling:

- iterative, tile/slab iterative, recursive,
- left/right looking, Crout

Search: defines the first r values of P and Q

Permutation: impacts all values of P and Q

Problem (Reformulation)

Under what conditions on the **Search** *and* **Permutation** *operations does a PLUQ decomposition algorithm reveals RowRP, ColRP or $\mathcal{R}^A$?*

The Pivoting matrix

Definition (The pivoting matrix)

Given a PLUQ decomposition $A = PLUQ$ with rank r, define

$$\Pi_{P,Q} = P \begin{bmatrix} I_r & \\ & \end{bmatrix} Q.$$

Locates the position of the pivots in the matrix A.

The Pivoting matrix

Definition (The pivoting matrix)

Given a PLUQ decomposition $A = PLUQ$ with rank r, define

$$\Pi_{P,Q} = P \begin{bmatrix} I_r & \\ & \end{bmatrix} Q.$$

Locates the position of the pivots in the matrix A.

Problem (Rank profile revealing PLUQ decompositions)

Under which conditions

- $\Pi_{P,Q} = \mathcal{R}^A$

The Pivoting matrix

Definition (The pivoting matrix)

Given a PLUQ decomposition $A = PLUQ$ with rank r, define

$$\Pi_{P,Q} = P \begin{bmatrix} I_r & \\ & \end{bmatrix} Q.$$

Locates the position of the pivots in the matrix A.

Problem (Rank profile revealing PLUQ decompositions)

Under which conditions

- $\Pi_{P,Q} = \mathcal{R}^A$
- $RowSupp(\Pi_{P,Q}) = RowSupp(\mathcal{R}^A) = RowRP(A)$ *(Weaker)*
- $ColSupp(\Pi_{P,Q}) = ColSupp(\mathcal{R}^A) = ColRP(A)$ *(Weaker)*

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

Search revealing rank profiles

- No stability issue over exact domains
- Intuition: must **minimize** some **ordering of the row/col indices** (notion of rank profile)

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

Search revealing rank profiles

- No stability issue over exact domains
- Intuition: must **minimize** some **ordering of the row/col indices** (notion of rank profile)

Example

Search: "Any non zero element on the topmost row":

$$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

Search revealing rank profiles

- No stability issue over exact domains
- Intuition: must **minimize** some **ordering of the row/col indices** (notion of rank profile)

Example

Search: "Any non zero element on the topmost row":

$$A = \begin{bmatrix} 2&0&3&0\\ 1&0&0&0\\ 0&0&4&0\\ 0&2&0&1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}, \quad \Pi_{P,Q} = \begin{bmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}.$$

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

Data locality: find pivot not too far from the main diagonal

Sparsity: find pivot that minimizes/reduce fill-in

Search revealing rank profiles

- No stability issue over exact domains
- Intuition: must **minimize** some **ordering of the row/col indices** (notion of rank profile)

Example

Search: "Any non zero element on the topmost row":

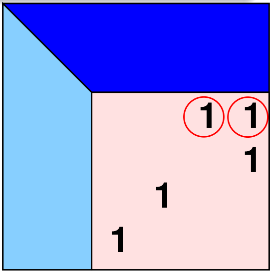
$$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Pi_{P,Q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \qquad \leadsto \text{RowRP}=\{1,2,4\}$$

Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row

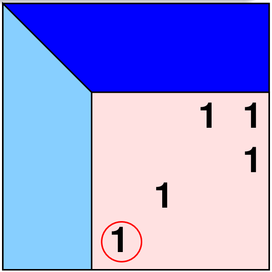


Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col



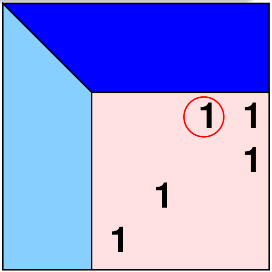
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex order: first non-zero on the first non-zero row



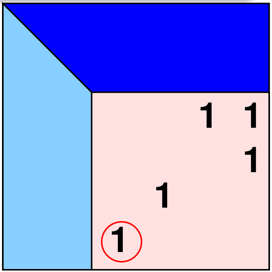
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex/RevLex order: first non-zero on the first non-zero row/col



Pivoting and permutation strategies

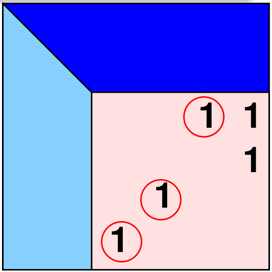
Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex/RevLex order: first non-zero on the first non-zero row/col

Product order: first non-zero in the (i, j) leading sub-matrix



Sufficient ?

Is lexicographic ordering sufficient to reveal both rank profiles?

Example

With a lexicographic ordering

1. $$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Pi_{P,Q}$$

Sufficient ?

Is lexicographic ordering sufficient to reveal both rank profiles?

Example

With a lexicographic ordering

1. $$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Pi_{P,Q}$$
2. But $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \rightsquigarrow \mathcal{R}^A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\Pi_{P,Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sufficient ?

Is lexicographic ordering sufficient to reveal both rank profiles?

Example

With a lexicographic ordering

1. $A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Pi_{P,Q}$
2. But $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \rightsquigarrow \mathcal{R}^A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\Pi_{P,Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$\rightsquigarrow$ Pivot Swaps mix-up precedence between rows/cols.

$\rightsquigarrow$ **Permutations** also have to be considered

Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

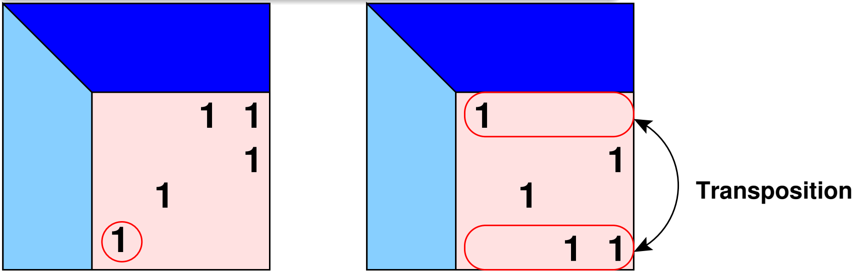
Row/Col order: any non-zero on the first non-zero row/col

Lex/RevLex order: first non-zero on the first non-zero row/col

Product order: first non-zero in the (i, j) leading sub-matrix

Permutation

- Transpositions



Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

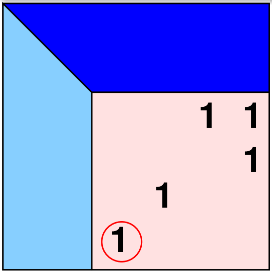
Row/Col order: any non-zero on the first non-zero row/col

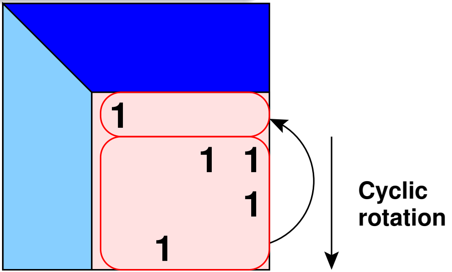
Lex/RevLex order: first non-zero on the first non-zero row/col

Product order: first non-zero in the (i, j) leading sub-matrix

Permutation

- Transpositions
- Cyclic Rotations





Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order Col. order						
Lexico.						
Rev. lex.						
Product						

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order Col. order	Transposition	Transposition	✓			[IMH82] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \left[\, 1\ 2\ \dots\ m \,\right] P \left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right]$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product						

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	Row perm.	Col. perm.	RowRP	ColRP	$\mathcal{R}^A$	Instance
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

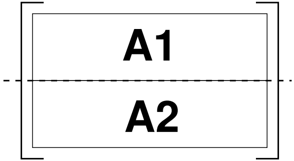
Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}^A$	**Instance**
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
Col. order	Transposition	Transposition		✓		[KG85] [JPS13]
Lexico.	Transposition	Transposition	✓			[Sto00]
Lexico.	Transposition	Rotation	✓	✓	✓	[DPS15]
Lexico.	Rotation	Rotation	✓	✓	✓	[DPS15]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Rev. lex.	Rotation	Transposition	✓	✓	✓	[DPS15]
Rev. lex.	Rotation	Rotation	✓	✓	✓	[DPS15]
Product	Rotation	Transposition	✓			[DPS15]
Product	Transposition	Rotation		✓		[DPS15]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

- RowRP $= \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- ColRP $= \begin{bmatrix} I_r & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 2 & \dots & m \end{bmatrix}^T$
- $\mathcal{R}^A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$

Outline

The slab recursive algorithm

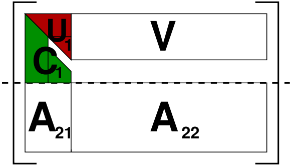
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise

The slab recursive algorithm

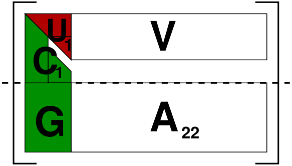
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1

The slab recursive algorithm

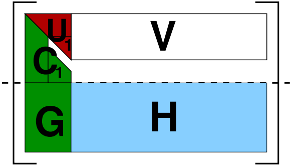
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)

The slab recursive algorithm

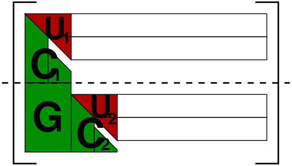
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)
4. $H \leftarrow A_{22} - G \times V$ (`MM`)

The slab recursive algorithm

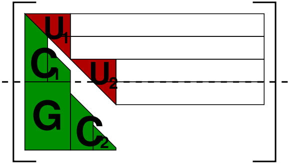
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)
4. $H \leftarrow A_{22} - G \times V$ (`MM`)
5. Recursive call on H

The slab recursive algorithm

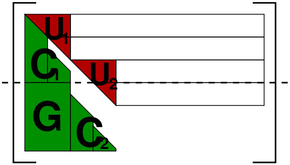
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)
4. $H \leftarrow A_{22} - G \times V$ (`MM`)
5. Recursive call on H
6. Row permutations

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



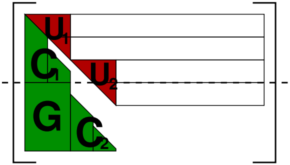
1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21} U_1^{-1}$ (`trsm`)
4. $H \leftarrow A_{22} - G \times V$ (`MM`)
5. Recursive call on H
6. Row permutations

Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



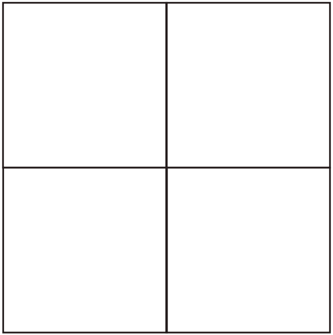
1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21} U_1^{-1}$ (`trsm`)
4. $H \leftarrow A_{22} - G \times V$ (`MM`)
5. Recursive call on H
6. Row permutations

Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP
- Row Rotations : Computes $\mathcal{R}^A$ [DPS15]

The tiled recursive algorithm

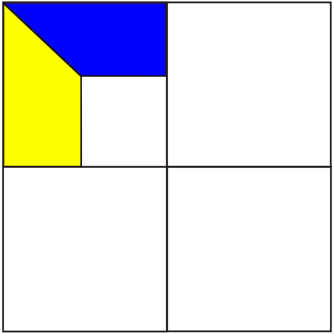
Dumas, P. and Sultan 13



2×2 block splitting

The tiled recursive algorithm

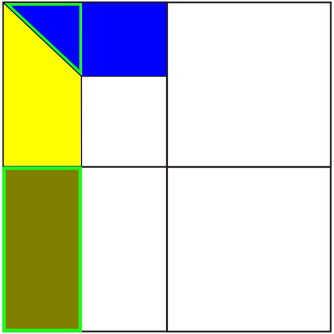
Dumas, P. and Sultan 13



Recursive call

The tiled recursive algorithm

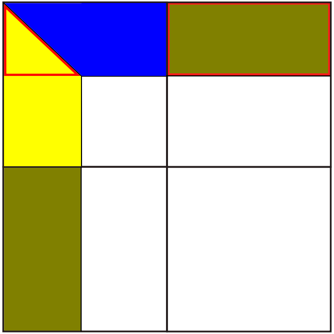
Dumas, P. and Sultan 13



TRSM: $B \leftarrow BU^{-1}$

The tiled recursive algorithm

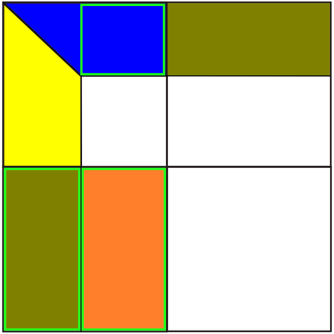
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

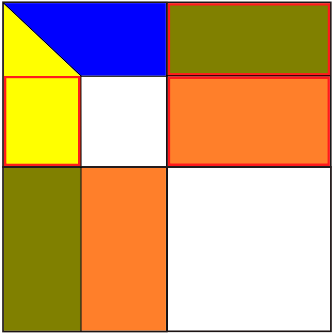
Dumas, P. and Sultan 13



$\texttt{MatMul}$: $C \leftarrow C - A \times B$

The tiled recursive algorithm

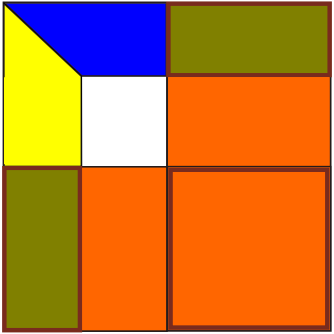
Dumas, P. and Sultan 13



`MatMul`: $C \leftarrow C - A \times B$

The tiled recursive algorithm

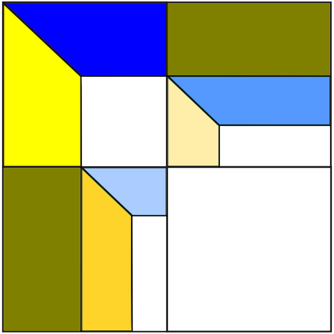
Dumas, P. and Sultan 13



`MatMul`: $C \leftarrow C - A \times B$

The tiled recursive algorithm

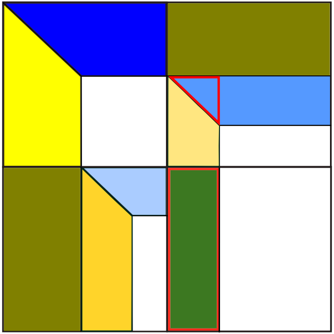
Dumas, P. and Sultan 13



2 independent recursive calls (compatible with the **product order**)

The tiled recursive algorithm

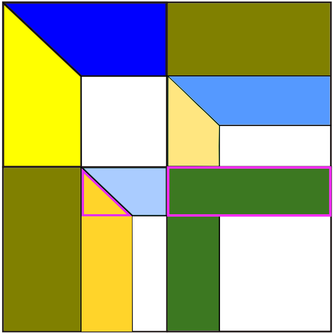
Dumas, P. and Sultan 13



TRSM: $B \leftarrow BU^{-1}$

The tiled recursive algorithm

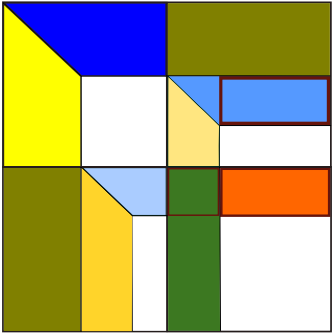
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

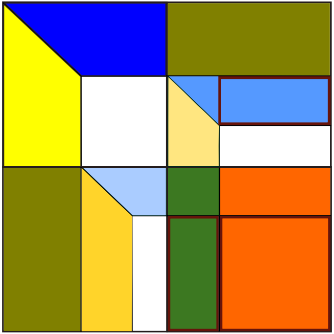
Dumas, P. and Sultan 13



MatMul: $C \leftarrow C - A \times B$

The tiled recursive algorithm

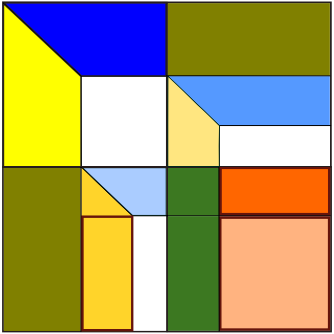
Dumas, P. and Sultan 13



MatMul: $C \leftarrow C - A \times B$

The tiled recursive algorithm

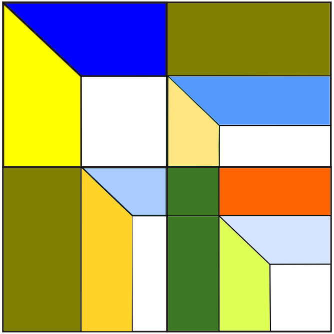
Dumas, P. and Sultan 13



`MatMul`: $C \leftarrow C - A \times B$

The tiled recursive algorithm

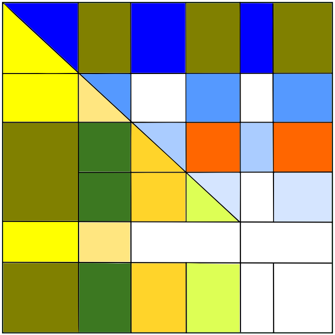
Dumas, P. and Sultan 13



Recursive call

The tiled recursive algorithm

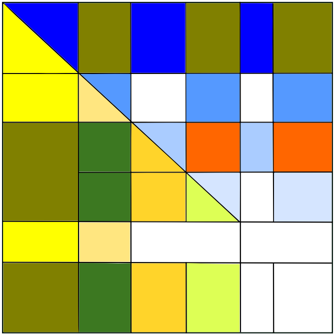
Dumas, P. and Sultan 13



Puzzle game (block **rotations**)

The tiled recursive algorithm

Dumas, P. and Sultan 13



- $O(mnr^{\omega-2})$ $(2/3n^3$ for $\omega = 3)$
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Iterative algorithms

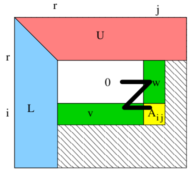
- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

Iterative algorithms

- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

Which base case algorithm?

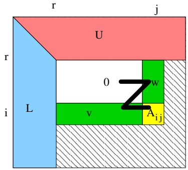
- Formerly [DPS13]: **product order** iterative algorithm
  - ✗ many permutations
  - ✗ many modular reductions



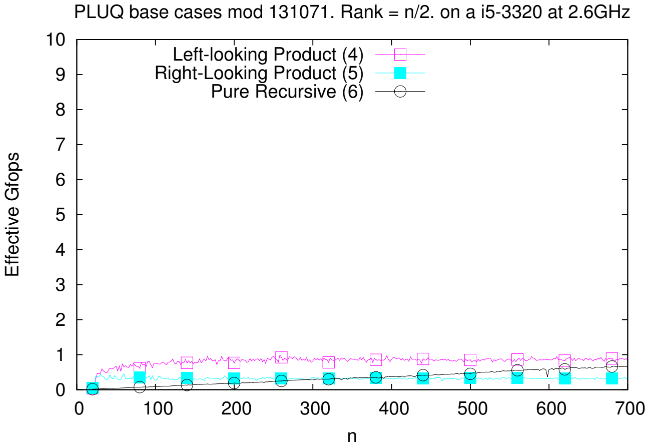
Iterative algorithms

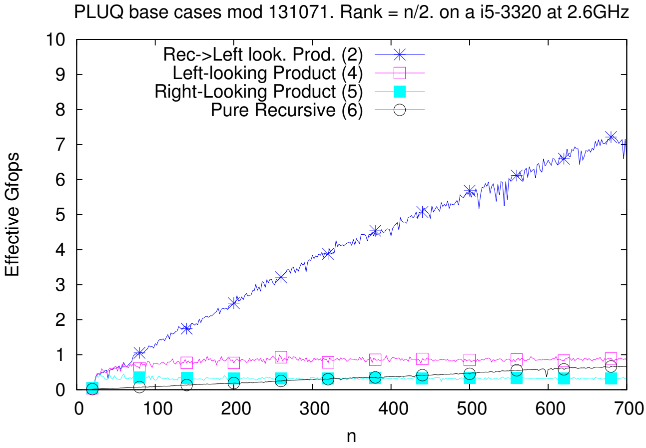
- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

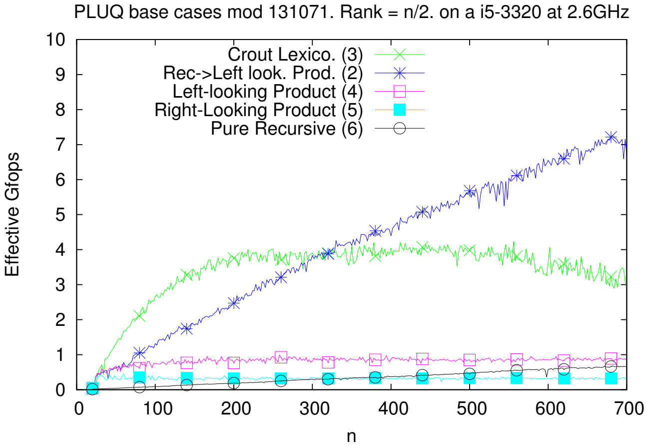
Which base case algorithm?

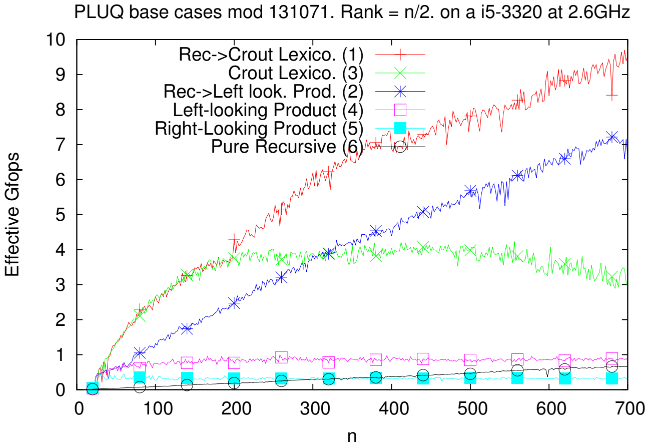


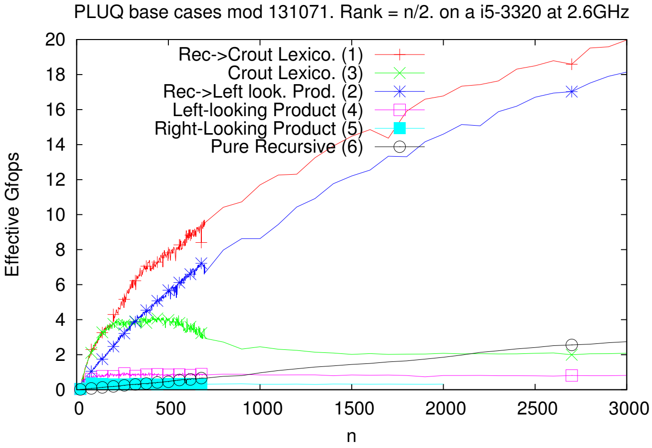
- Formerly [DPS13]: **product order** iterative algorithm
  - ✗ many permutations
  - ✗ many modular reductions
- [DPS15]: Simply use the schoolbook algorithm (Lexico+Rotations)
  - ✓ fewer permutations
  - ✓ modular reductions delayed more easily
  - ✓ Crout variant: better data access pattern

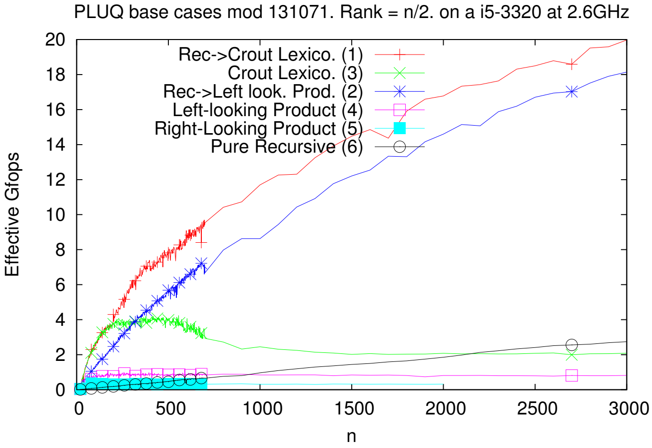

PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
Effective Gfops
n
0
1
2
3
4
5
6
7
8
9
10
0
100
200
300
400
500
600
700


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
Effective Gfops
n
0
1
2
3
4
5
6
7
8
9
10
0
100
200
300
400
500
600
700


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Effective Gfops
n
Crout Lexico. (3)
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
0
1
2
3
4
5
6
7
8
9
10
0
100
200
300
400
500
600
700


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Rec->Crout Lexico. (1)
Crout Lexico. (3)
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
Effective Gfops
n
0
1
2
3
4
5
6
7
8
9
10
100
200
300
400
500
600
700


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Effective Gfops
n
Rec->Crout Lexico. (1)
Crout Lexico. (3)
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
0
2
4
6
8
10
12
14
16
18
20
0
500
1000
1500
2000
2500
3000



- > 2 Gfops improvement
- Implemented in `FFLAS-FFPACK` (kernel of `LinBox`).

Outline

Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^\omega - 4} MM(m, n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

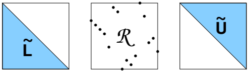
Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^{\omega}-4} MM(m,n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

Fact

$$E = \mathcal{R}^A$$



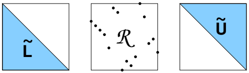
Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^\omega - 4} MM(m,n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

Fact

$$E = \mathcal{R}^A$$



$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

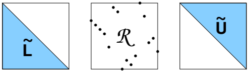
Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^{\omega}-4} MM(m,n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

Fact

$$E = \mathcal{R}^A$$



$$A = PLUQ = P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} QQ^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

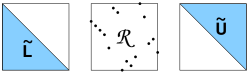
Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^\omega - 4} MM(m,n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

Fact

$$E = \mathcal{R}^A$$



$$A = PLUQ = \underbrace{P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T}_{\overline{L}} \underbrace{P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q}_{\Pi_{P,Q}} \underbrace{Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q}_{\overline{U}}$$

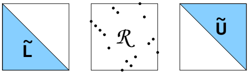
Malaschonok LEU decomposition

[Malaschonok'10]: $A = L \cdot E \cdot U$

- E is an r-sub-permutation matrix
- Designed to avoid permutations
- $\frac{17}{2^\omega - 4} MM(m,n)$ with $m = n = 2^k$.
- no connection to rank profile nor echelon form
- no rank sensitive complexity

Fact

$$E = \mathcal{R}^A$$



$$A = PLUQ = \underbrace{P \begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} P^T}_{\overline{L}} \underbrace{P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q}_{\Pi_{P,Q}} \underbrace{Q^T \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q}_{\overline{U}}$$

With appropriate pivoting: $\Pi_{P,Q} = \mathcal{R}(A)$

LUP and PLU decompositions

LUP

If A has generic RowRP

- $LUP(A)$ with Lex order and col. rot.: $\rightsquigarrow \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} P = \mathcal{R}^A$

In particular, if A has full row rank and $m = n$: $\rightsquigarrow P = \mathcal{R}^A$

LUP and PLU decompositions

LUP

If A has generic RowRP

- $LUP(A)$ with Lex order and col. rot.: $\rightsquigarrow \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} P = \mathcal{R}^A$

In particular, if A has full row rank and $m = n$: $\rightsquigarrow P = \mathcal{R}^A$

PLU

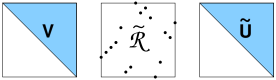
If A has generic ColRP

- $PLU(A)$ with RevLex order and row rot. $\rightsquigarrow P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} = \mathcal{R}^A$

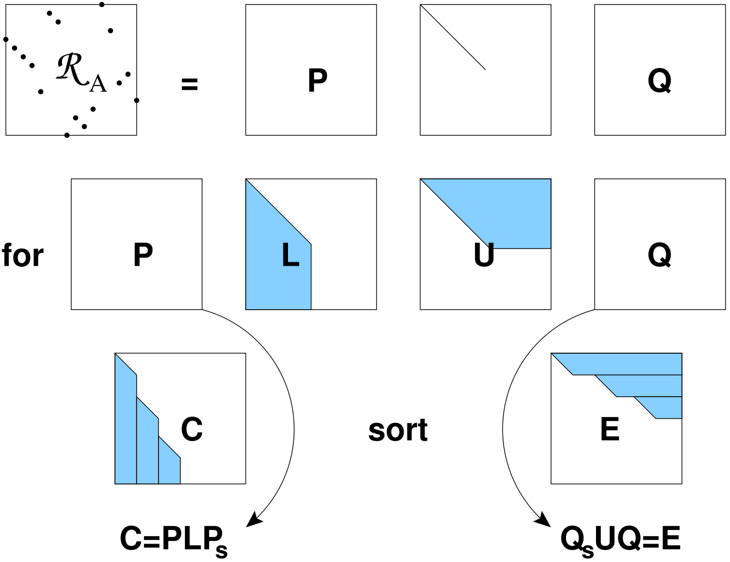
In particular, if A has full column rank and $m = n$: $\rightsquigarrow P = \mathcal{R}^A$

Bruhat decomposition

- If $A = \tilde{L}\mathcal{R}^A\tilde{U}$, then
  - For J_n the unit anti-diagonal matrix,
  - $V = J_n\tilde{L}J_n$ is upper triangular
  - $\tilde{\mathcal{R}} = J_n\mathcal{R}^A$ is a rank r sub-permutation
  - $A = V\tilde{\mathcal{R}}\tilde{U}$ (Bruhat decomposition)



Echelon forms



Outline

Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
(Compressed sensing applications)

[Cheung Kwok Lau'12]: Compute the rank r and r linearly independent rows in $\tilde{O}(r^\omega + mn)$ probabilistic

Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
(Compressed sensing applications)

[Cheung Kwok Lau'12]: Compute the rank r and r linearly independent rows in $\tilde{O}(r^\omega + mn)$ probabilistic

[Storjohann Yang'14:] Rank profile in $\tilde{O}(r^3 + mn)$ probabilistic.

Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
(Compressed sensing applications)

[Cheung Kwok Lau'12]: Compute the rank r and r linearly independent rows in $O\tilde{~}(r^{\omega} + mn)$ probabilistic

[Storjohann Yang'14:] Rank profile in $O\tilde{~}(r^3 + mn)$ probabilistic.

[Storjohann Yang'15:] Rank profile in $O\tilde{~}(r^{\omega} + mn)$ probabilistic.

Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
(Compressed sensing applications)

[Cheung Kwok Lau'12]: Compute the rank r and r linearly independent rows in $O\tilde{}(r^{\omega} + mn)$ probabilistic

[Storjohann Yang'14:] Rank profile in $O\tilde{}(r^3 + mn)$ probabilistic.

[Storjohann Yang'15:] Rank profile in $O\tilde{}(r^{\omega} + mn)$ probabilistic.

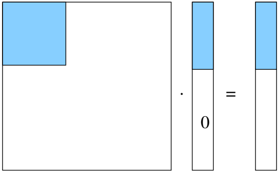
Can the rank profile matrix be computed in such complexities?

[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.



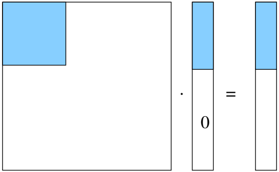
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$



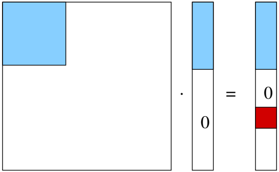
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$



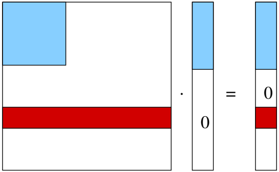
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$



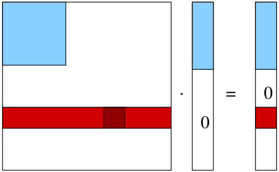
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$



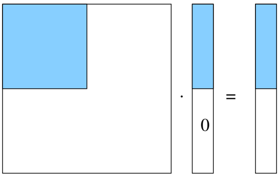
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$
2. Compute A_{s+1}^{-1} by rank 1 updates $\rightsquigarrow O(s^2)$



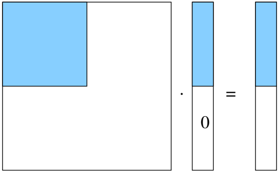
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\mathrm{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(sn)$
2. Compute A_{s+1}^{-1} by rank 1 updates $\rightsquigarrow O(s^2)$



- Use the vector b to compress row linear dependency information

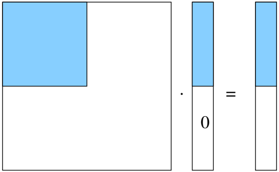
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(s \log n)$
2. Compute A_{s+1}^{-1} by rank 1 updates $\rightsquigarrow O(s^2)$



- Use the vector b to compress row linear dependency information
- Improved by linear independence oracles

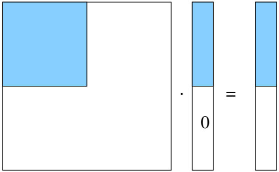
[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.

1. Use A_s^{-1} to find the next row and column to append to A_s. $\rightsquigarrow O(s \log n)$
2. Compute A_{s+1}^{-1} by rank 1 updates $\rightsquigarrow O(s^2)$



- Use the vector b to compress row linear dependency information
- Improved by linear independence oracles

Lexico. search with rotations $\rightsquigarrow$ computes $\mathcal{R}^A$

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $\tilde{O}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively $(O(r^3))$, use an asymptotically fast relaxation scheme $O(r^\omega)$.
2. Requires to deal with only r columns in generic column RP.
3. Ensured by a call to [Cheung Kwok Lau'12] $+$ Toeplitz preconditionner
4. Returns the row rank profile

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $O\tilde{}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
2. Requires to deal with only r columns in generic column RP.
3. Ensured by a call to [Cheung Kwok Lau'12] $+$ Toeplitz preconditionner
4. Returns the row rank profile

Problem: step 3 loses information required for the $\mathcal{R}^A$.

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $\tilde{O}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
2. Requires to deal with only r columns in generic column RP.
3. Ensured by a call to [Cheung Kwok Lau'12] $+$ Toeplitz preconditionner
4. Returns the row rank profile

Problem: step 3 loses information required for the $\mathcal{R}^A$.

Solution for $\mathcal{R}^A$ in $\tilde{O}(r^\omega + mn)$

1. Compute the RowRP $\mathcal{I}$ by [Storjohann Yang'15] on A
2. Compute the ColRP $\mathcal{J}$ by [Storjohann Yang'15] on A^T

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $O\tilde{}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
2. Requires to deal with only r columns in generic column RP.
3. Ensured by a call to [Cheung Kwok Lau'12] $+$ Toeplitz preconditionner
4. Returns the row rank profile

Problem: step 3 loses information required for the $\mathcal{R}^A$.

Solution for $\mathcal{R}^A$ in $O\tilde{}(r^\omega + mn)$

1. Compute the RowRP $\mathcal{I}$ by [Storjohann Yang'15] on A
2. Compute the ColRP $\mathcal{J}$ by [Storjohann Yang'15] on A^T
3. Extract the $r \times r$ submatrix $A_r = A_{\mathcal{I},\mathcal{J}}$
4. Compute the LUP decomp of A_r with col. rotations

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $O\tilde{}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
2. Requires to deal with only r columns in generic column RP.
3. Ensured by a call to [Cheung Kwok Lau'12] $+$ Toeplitz preconditionner
4. Returns the row rank profile

Problem: step 3 loses information required for the $\mathcal{R}^A$.

Solution for $\mathcal{R}^A$ in $O\tilde{}(r^\omega + mn)$

1. Compute the RowRP $\mathcal{I}$ by [Storjohann Yang'15] on A
2. Compute the ColRP $\mathcal{J}$ by [Storjohann Yang'15] on A^T
3. Extract the $r \times r$ submatrix $A_r = A_{\mathcal{I},\mathcal{J}}$
4. Compute the LUP decomp of A_r with col. rotations
5. Recover $\mathcal{R}^A$ by inflating $\mathcal{R}^{A_r} = P$ with zeroes.

Perspective

- Application to F5 elimination (Gröbner basis) [Sun Lin Wang'14]
- Communication avoiding variants [Demmel & Al.'12]
- How to accomodate sparse elimination constraints ?
- Numerical pivoting equivalent?

Perspective

- Application to F5 elimination (Gröbner basis) [Sun Lin Wang'14]
- Communication avoiding variants [Demmel & Al.'12]
- How to accomodate sparse elimination constraints ?
- Numerical pivoting equivalent?

Thank you!

Bibliography

[Malaschonok'10]: $A = LEU$

- first instance of $\mathcal{R}^A$.
- no consideration on rank profile nor echelon form

[DSP'13]: $A = PLUQ$

Computed only via a product order pivoting,
Rank sensitive $O(r^{\omega-2}mn)$, any $m \times n$ matrix of any rank r.

[DPS'15]

- Conditions for any PLUQ alg. to reveal $\mathcal{R}^A$
- New pivoting strategies $\rightsquigarrow$ faster base case

[DPS'XX in preparation]

- $\mathcal{R}^A$ in $\tilde{O}(r^\omega + mn)$
- generalization of $\mathcal{R}^A$ to rings