Proof of a KT4 prover

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Abstract. We present a new proof of a decision procedure for the propositional modal logic KT4. This procedure indicates if a formula is contradictory or gives a finite model of a satisfiable formula. It is inspired by a unified environment for various kinds of modal operators described by Laurent Catach . His general method gives indications on rules to detect looping, which are necessary whenever transitivity is considered. We precise these rules and prove their correctness. The approach extends well to multimodal languages with transitive operators.

Introduction

It is well known that the modal logic KT4 is decidable [3, 11]. A formula A of this logic is satisfiable if and only if it is satisfiable by a reflexive, transitive, pointed valued structure whose domain contains at most 2^n elements, where n is the number of subformulas of A. This theoretical result provides a decision procedure that is totally unrealistic as it requires the evaluation of the formula on all these valued structures. This prohibitive cost motivated the decision procedure described here, which follows the principles set forth by Laurent Catach [2]. It can be observed that the general method described in [2] contains imprecisions in the manner in which the unwinding test (see Remark 2) is set forth and an uncontrolled application of this test can lead to contradictions that are absent from the original formula.

Our objective is to present the precise rules for this method so that to avoid these problems. After introducing preliminary definitions, we describe the proposed decision procedure. Then we show its coherence and its completeness. A prover based on this method has been implemented and its preformances are described in [8].

1 Preliminaries

1.1 Syntax

In order to simplify the presentation, we limit the logical connectives to be negation, conjunction, necessary denoted as $\neg, \wedge, \Box$. This set of connectives is complete for the modal logic KT4.

Definition 1 (Hypothesis).
A hypothesis is a pair consisting of an integer i and a formula A that is denoted by $i : A$. The integer i is said to be the state *of the hypothesis $i : A$ and the formula A is its formula. A hypothesis has the following intuitive meaning : the formula A is true in state i.*

Definition 2 (Edge).
An edge is a pair of integers. Two kinds of edges are considered : the simple *edges, which are denoted by $i \rightarrow j$, and the* double *edges, which are denoted by $i \twoheadrightarrow j$ where i, j are integers.*

Definition 3 (Paths and Tableaux).
A path is a list of edges and hypotheses.
A tableau is a list of paths.

1.2 Semantics

Definition 4 (Interpretation, valued structure).
Let X be a set of variables.

- *an interpretation for X is a 3-tuple (D, R, v), where*
  - *D is a non-empty domain*
  - *R is a* reflexive, transitive *relation on D*
  - $v : X \rightarrow 2^D isamappingfromXintothesubsetsofD$
- *A* pointed *valued structure for X is a 4-tuple (D, R, v, e), where (D, R,v) is an interpretation for X and e is an element of D.*
- *A* global *valued structure for X is a 4-tuple (D, R, v, h) where (D, R,v) is an interpretation for X and h is a application of $\mathbb{N}^*$ in D.*

Let I be (D, R, v) an interpretation for X. The mapping v_I denotes the extension *of v to the formulas whose variables are elements of X. This extension is defined by :*

- $v_I(\Box A) = \{e \in D | \forall f \in D, eRf \Rightarrow f \in v_I(A)\}$
- $v_I(A \bigcap B) = v_I(A) \bigcap v_I(B)$
- $v_I(\neg A) = D - v_I(A)$

When the context is unambiguous, we will omit the index I in v_I.

Definition 5 (Satisfaction of formulas).
Let X be a set of variables. Let α = (D, R, V, e) a pointed valued structure for X and let A be a formula on X. α satisfies A if and only if e is an element of $v(A)$.

Definition 6 (Satisfaction of *simple* edges, hypotheses, formulas and tableaux).
Let α = (D, R, V, h) a global valued structure.

- *α satisfies the* simple *edge $i \rightarrow j$ if and only if $h(i)$ R $h(j)$.*

– *α satisfies the hypothesis $i : A$ if and only if $h(i) \in v(A)$.*
– *α satisfies a path (which is a list of edges and hypotheses) if and only if α satisfies the* simple *edges of the path and the hypotheses of the path.*
– *α satisfies a tableau if and only if α satisfies one of the paths of the tableau.*

Definition 7 (Satisfiability).

– *A path is satisfiable if and only if there is a global valued structure which satisfies it*
– *A tableau is satisfiable if and only if there is a global valued structure which satisfies it.*

Note that in the above definitions, we have omitted the set of variables used. We use an implicit assumption that the valued structure has defined at least the values of the variables of the formulas we wish to evaluate. This omission extends to the remainder of this paper.

2 Rules governing tableaux

We introduce the rules, first by an example and then by an detailed presentation in 2.3.

2.1 Preliminary definitions

Definition 8 (Closed path and tableau).
Two hypotheses are complementary if one is of the form $i : A$ and the other is of the form $i : \neg A$.
A path is closed if there are two complementary hypotheses on it. A tableau is closed if all its paths are closed.

Definition 9 (Final path and tableau).
The hypotheses can be marked : a mark on a hypothesis indicates that it has been used.
An elementary hypothesis is a hypothesis of the form $i : A$ where A is a literal.
A path is final if it is a non-closed path and if all its hypotheses are either marked or elementary. A tableau is final if it has a final path.

Definition 10 (Universal, existential hypothesis).
A universal hypothesis is a hypothesis of the form $i : \Box A$. An existential hypothesis is a hypothesis of the form $i : \neg\Box A$.

2.2 Example

First we give an example of the application of the proposed method.

Let A be the formula $\Box\Diamond p \Rightarrow \Box p$. We show that A is not valid. In the first place we transform the $\neg A$ by using the definition of $\Diamond$ and $\Rightarrow$. In the second place we use the tableau method described below to produce a pointed valued structure satisfying $\neg A$.

In this example, we do not use any alpha-rule (see Alpha-rule in 2.3). Hence all tableaux will have only one path. They will be represented by a list of edges and hypotheses.

Already used hypotheses are underlined except the last used which is underbraced.

1. $1 : \Box\neg\Box\neg p \wedge \neg\Box p$
2. $\underbrace{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $1 : \Box\neg\Box\neg p$, $1 : \neg\Box p$

   The rule $\wedge$ adds the hypotheses $1 : \Box\neg\Box\neg p$, $1 : \neg\Box p$.
3. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underbrace{1 : \Box\neg\Box\neg p}$, $1 : \neg\Box p$, $1 : \neg\Box\neg p$

   The rule $\Box$ adds the hypothesis $1 : \neg\Box\neg p$.
4. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underbrace{1 : \neg\Box p}$, $1 : \neg\Box\neg p$, $1 \rightarrow 2$, $2 : \neg p$, $2 : \Box\neg\Box\neg p$

   The rule $\neg\Box$ creates the new state 2, adds the edge $1 \rightarrow 2$ and the hypotheses $2 : \neg p$, $2 : \Box\neg\Box\neg p$.
5. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $1 : \neg\Box\neg p$, $1 \rightarrow 2$, $2 : \neg p$, $\underbrace{2 : \Box\neg\Box\neg p}$, $2 : \neg\Box\neg p$

   The rule $\neg\Box$ cannot be applied, but can be applied the rule $\Box$. It adds the hypothesis $2 : \neg\Box\neg p$.
6. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $\underbrace{1 : \neg\Box\neg p}$, $1 \rightarrow 2$, $2 : \neg p$, $\underline{2 : \Box\neg\Box\neg p}$, $2 : \neg\Box\neg p$, $1 \rightarrow 3$, $3 : \neg\neg p$, $3 : \Box\neg\Box\neg p$

   The rule $\neg\Box$ creates the state 3, adds the edge $1 \rightarrow 3$ and the hypotheses $3 : \neg\neg p$, $3 : \Box\neg\Box\neg p$.
7. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $\underline{1 : \neg\Box\neg p}$, $1 \rightarrow 2$, $2 : \neg p$, $\underline{2 : \Box\neg\Box\neg p}$, $2 : \neg\Box\neg p$, $1 \rightarrow 3$, $\underbrace{3 : \neg\neg p}$, $3 : \Box\neg\Box\neg p$, $3 : p$

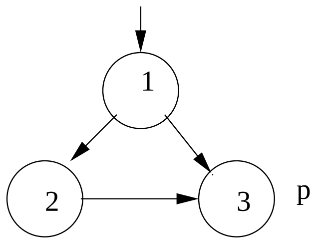
   The rule $\neg\Box$ can not be applied, why we can apply the rule $\neg\neg$ which adds the hypothesis $3 : p$.
8. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $\underline{1 : \neg\Box\neg p}$, $1 \rightarrow 2$, $2 : \neg p$, $\underline{2 : \Box\neg\Box\neg p}$, $2 : \neg\Box\neg p$, $1 \rightarrow 3$, $\underline{3 : \neg\neg p}$, $\underbrace{3 : \Box\neg\Box\neg p}$, $3 : p$, $3 : \neg\Box\neg p$

   The rule $\neg\Box$ can not be applied, why we can apply the rule $\Box$ which adds the hypothesis $3 : \neg\Box\neg p$.
9. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $\underline{1 : \neg\Box\neg p}$, $1 \rightarrow 2$, $2 : \neg p$, $\underline{2 : \Box\neg\Box\neg p}$, $\underbrace{2 : \neg\Box\neg p}$, $1 \rightarrow 3$, $\underline{3 : \neg\neg p}$, $\underline{3 : \Box\neg\Box\neg p}$, $3 : p$, $3 : \neg\Box\neg p$, $2 \twoheadrightarrow 3$

   The rule $\neg\Box$ is applied and adds the double edge $2 \twoheadrightarrow 3$.
10. $\underline{1 : \Box\neg\Box\neg p \wedge \neg\Box p}$, $\underline{1 : \Box\neg\Box\neg p}$, $\underline{1 : \neg\Box p}$, $\underline{1 : \neg\Box\neg p}$, $1 \rightarrow 2$, $2 : \neg p$, $\underline{2 : \Box\neg\Box\neg p}$, $\underline{2 : \neg\Box\neg p}$, $1 \rightarrow 3$, $\underline{3 : \neg\neg p}$, $\underline{3 : \Box\neg\Box\neg p}$, $3 : p$, $\underbrace{3 : \neg\Box\neg p}$, $2 \twoheadrightarrow 3$

    The rule $\neg\Box$ is applied and adds nothing : we have a final path.

From the final path, we extract the pointed valued structure ({1,2,3}, R, v, 1) represented below. We have v(p) = 3 and R is the reflexive and transitive closure of the relation represented by the arrows. It is clear that this structure is a *counter-model* of the formula $\Box\Diamond p \Rightarrow \Box p$.



2.3 Paths and tableau relations

The rules governing the tableaux are defined by means of relations between paths which are extended to the tableaux.

Definition 11 (Largest state of a path). *Let Γ be a path and let n be the largest integer occurring in the states of the hypotheses appearing on the path Γ. We say that n is the largest state of Γ.*

Definition 12 (Set of formulas associated to a state). *Let Γ be a path whose largest state is n. For i from 1 to n, the set Γ_i of formulas of the path Γ, which are in the state i, is defined by : $\Gamma_i = \{A \mid$ the hypothesis $i : A$ is on Γ $\}$.*

Alpha-rule. Let Γ be a path whose largest state is n. The path Γ is an alpha-predecessor of the paths Γ' and Γ'' in the following case :

rule $\neg\wedge$: the path Γ has a non-marked hypothesis of the form $n : \neg(A \wedge B)$, and Γ' (resp. Γ'') is a copy of the path Γ in which this hypothesis is marked and to which the hypothesis $n : \neg A$ (resp. $n : \neg B$) has been added.

Beta-rule. Let Γ be a path whose largest state is n. The path Γ is beta-predecessor of the path Γ' if one of the four conditions below holds

rule $\wedge$: (see Example 2.2 line 2) the path Γ has a non-marked hypothesis of the form $n : (A \wedge B)$, and Γ' is a copy of the path Γ in which this hypothesis is marked and to which the two hypotheses $n : A$ and $n : B$ have been added.

rule $\neg\neg$: (see Example 2.2 line 7) the path Γ has a non-marked hypothesis of the form $n : \neg\neg A$, and Γ' is a copy of the path Γ in which this hypothesis is marked and to which the hypothesis $n : A$ has been added.

rule $\Box$: (see Example 2.2 line 3) the path Γ has a non-marked hypothesis of the form $n : \Box A$, and Γ' is a copy of the path Γ in which this hypothesis is marked and to which the hypothesis $n : A$ has been added.

rule $\neg\Box$: the path Γ has a non-marked hypothesis of the form $i : \neg\Box A$ ($1 \leq i \leq n$) and *all non existential hypotheses are marked or elementary* [1].

[1] The formula of existential hypothesis begins with $\neg\Box$ and the formula of an elementary hypothesis is a litteral. The rule $\neg\Box$ is the only one which can be applied to hypotheses whose state is not the largest state of Γ (see Example 2.2 line 6) and it can be applied only if the previous rules can not be applied (see Example 2.2 line 5).

- If there exists a integer k between 1 and n such that
  - the hypothesis $k : \neg A$ is contained by Γ
  - for every hypothesis of the form $i : \Box B$ which is contained by Γ, the hypothesis $k : \Box B$ is also contained by Γ

  then (see Example 2.2 lines 9 and 10) Γ' is a copy of the path Γ in which the hypothesis $i : \neg\Box A$ is marked and to which the *double* edge $i \twoheadrightarrow k$ has been added only if $i \neq k$.
- If such an integer does not exist, then (see Example 2.2 line 4) Γ' is a copy of the path Γ in which the hypothesis $i : \neg\Box A$ is marked and to which one adds
  - the *simple* edge $i \rightarrow n+1$ (in this case, we say that we have created the new state n+1)
  - the hypothesis $n+1 : \neg A$
  - all the hypotheses $n+1 : \Box B$ for every hypothesis of the form $i : \Box B$ which is on the path Γ

Remark 1. From a algorithmic point of view, the distinction between simple and double edges is unnecessary. This distinction only helps in the proof of the correctness of the method.

Remark 2 (unwinding test). The rule $\neg\Box$ applied to a path Γ whose largest state is n, does not create a new state n+1, if that creation would produce a path Γ' with Γ'_{n+1} included in some Γ'_i where $1 \leq i \leq n$. This condition is called the *unwinding test* in [2].

Successor of a tableau. Let T be a non-empty and non-final tableau. The tableau T' is a successor of T, if T' is obtained

- either by replacing a path Γ having two alpha-predecessors Γ' and Γ'' by Γ' (if Γ' is not closed) and Γ'' (if Γ'' is not closed).
- or by replacing a path Γ having a beta-predecessor Γ' by Γ' (if Γ' is not closed).

2.4 Tableau procedure

Definition 13. *A tableau T is constructed from a formula A, if there is a sequence $T_{i(0 \leq i \leq n)}$ such that*

- *T_0 has only one path I(A) having the unique hypothesis $1 : A$*
- *T is the tableau T_n*
- *for all $i < n$, the tableau T_{i+1} is a successor of T_i for the successor relation defined above*

To decide if a formula A is satisfiable or not, we built a sequence of tableaux whose first term is the tableau which has only one path I(A), whose each following term is obtained by applying a rule. We stop as soon as we obtain an empty tableau (A is declared unsatisfiable) or a final tableau (A is declared satisfiable).

We will see in the next section that this procedure gives the right answers and stops.

3 Correctness proof

We prove successively that :

- if the empty tableau is constructed from a formula , this formula is unsatisfiable
- if a final tableau is constructed from a formula , this tableau has a final path to which we can associate a pointed valued structure which satisfies the formula
- there is no infinite sequence $T_{i(0 \leq i)}$ of tableaux constructed from a formula, such that T_{i+1} is a successor of T_i

3.1 Consistency proof

Property 1. If a path is satisfiable then any predecessor of this path is also satisfiable.

Proof. Whe show this result for a single case, that one of the rule $\neg\Box$ (which is the paradigmatic case).
Let Γ a satisfiable path. Let n be the largest state of Γ.
There exists a global valued structure α = (D, R, v, h) which satisfies Γ.
Let Γ' be a beta-predecessor of the path Γ for the rule $\neg\Box$.
We consider two cases :

- If no state has been created, Γ and Γ' have the same hypotheses and the same *simple* edges. Since the double edges play no role in the satisfaction of a path (see Definition 6), Γ' is also satisfiable .
- If a state n+1 has been created, Γ has an non-marked hypothesis $i : \neg\Box A$, Γ' is a copy of Γ in which this hypothesis is marked and contains the new *simple* edge $i \rightarrow n+1$, the new hypothesis $n+1 : \neg A$ and the new hypotheses $n+1 : \Box B$ for every hypothesis $i : \Box B$ which is on Γ.
  Since the global valued structure α satisfies Γ, it satisfies $i : \neg\Box A$, and by Definition 6 we have $h(i) \in v(\neg\Box A)$. Consequently there exists e in D, such that
  1. $h(i) R\ e$
  2. $e \in v(\neg A)$

  Let h' be h[e\n+1] and let α' be (D, R, v , h').
  Since α and α' differ only on the new state, α' satisfies Γ. In order to prove that α' satisfies Γ', it is enough to prove that α' satisfies the new elements added to Γ'.
  - The edge $i \rightarrow n+1$ is satisfied. As $i \neq n+1$, by definition of h', h'(i) = h(i). We have h'(n+1) = e and by (1), we have h'(i) R h'(n+1).
  - The hypothesis $n+1 : \neg A$ is satisfied. From (2) and the definition of h', we have $h'(n+1) \in v(\neg A)$
  - Assume that the hypothesis $i : \Box B$ is on Γ. Since $h(i) \in v(\Box B)$, and since h(i) R e and since R is a *reflexive and transitive* relation, we have $e \in v(\Box B)$. Consequently, $h'(n+1) \in v(\Box A)$.

$\square$

Theorem 1 (Consistency). *If the empty tableau is constructed from a formula, this formula is unsatisfiable*

Proof. Let A be a satisfiable formula. There exists a pointed valued structure (D, R, v, e) such that $e \in v(A)$. The global valued structure (D, R, v, h) such that h(1) = e satisfies the path I(A), which has the unique hypothesis $1 : A$. From Property 1, from Definition 13 and from the fact that the path I(A) is satisfiable, we deduce easily that every tableau constructed from A has a satisfiable path.

Consequently, if the empty tableau is constructed from the formula A, this formula is unsatisfiable. $\square$

3.2 Construction of a model of a final path

We define what is a complete path (see below Definition 15) and we show, by the Lemma 1, that every path built by the tableau method is complete.

Definition 14 (Normal path). *Let Γ be a path and let n be the largest state of Γ. Γ is normal if*

1. *The state of all non-existential, non-elementary and non-marked hypotheses is n.*
2. *If there is on Γ an* marked *existential hypothesis of the form $n : \neg\Box A$, then all non-existential hypothesis are marked or elementary.*

Property 2. If a path is normal, then any predecessor of this path is normal.

Proof. Assume that the path Γ is normal. Let n be the largest state of Γ and let Γ' be a predecessor of Γ.
We consider the three following cases :

- If Γ' is obtained by one of the rules, except $\neg\Box$, the path Γ' is a copy of Γ in which an *non-existential hypothesis of state n is newly marked* and to which new hypotheses of state n are added. Consequently Γ' is normal.
- If Γ' is obtained by the rule $\neg\Box$, without creation of a new state, Γ' is a copy of Γ in which an existential hypothesis of state i ($1 \leq i \leq n$) is newly marked and all non-existential hypotheses are already marked or elementary. Consequently Γ' is normal.
- If Γ' is obtained by the rule $\neg\Box$, with creation of the state n+1, the new hypotheses added to Γ' are of state n+1, the largest state of Γ'. Consequently Γ' is normal.

$\square$

Property 3. The paths of a tableau constructed from a formula, are normal.

Proof. Immediate from Property 2. $\square$

Definition 15 (Complete path). *A path is complete if it verifies the following conditions :*

(c1) *If the hypothesis $i : \neg(A \wedge B)$ is marked on the path, then one of two hypotheses $i : \neg A$, $i : \neg B$ is on the path.*

(c2) *If the hypothesis $i : (A \wedge B)$ is marked on the path, then the two hypotheses $i : A$ and $i : B$ are on the path.*

(c3) *If the hypothesis $i : \neg\neg A$ is marked on the path, then the hypothesis $i : A$ is on the path.*

(c4) *If the hypothesis $i : \Box A$ is marked on the path, then the hypothesis $i : A$ is on the path and, for every* simple *edge $i \rightarrow j$ which is on the path and for every* double *edge $i \twoheadrightarrow j$ which is on the path, all the hypotheses $j : \Box A$ are on the path*

(c5) *If the hypothesis $i : \neg\Box A$ is marked on the path, then there exists an integer j such that the hypothesis $j : \neg A$ is on the path and if $i \neq j$, there is on the path either a simple edge $i \rightarrow j$ or a double edge $i \twoheadrightarrow j$.*

Property 4. If a path is normal and complete then any predecessor of this path is complete.

Proof. Let Γ be a path normal and complete. Let n be the largest state of Γ and let Γ' be a predecessor of Γ.
We consider two cases :

- If Γ' is obtained by one of the rules, except $\Box$, the path Γ' is clearly complete.
- If Γ' is obtained by the rule $\Box$, then Γ has an non-marked hypothesis of the form $n : \Box A$ and Γ' is a copy of Γ in wich this hypothesis is marked and to wich the hypothesis $n : A$ has been added. Since Γ is normal (Definition 14), no hypothesis of the form $n : \neg\Box B$ is marked on Γ. Consequently, no edge is going out of n neither on Γ or on Γ'. Since there is no edge going out of n on Γ', this path verifies the condition (c4) of Definition 15 and trivially the other conditions of this Definition because they were already assumed by Γ. Consequently Γ' is complete.

$\Box$

Lemma 1 (All paths are complete). *The paths of a tableau constructed from a formula, are complete.*

Proof. Immediate from Property 3 and Property 4 $\Box$

Definition 16 (Global valued structure associated to a path). *Let Γ be a path and let X be the set of variables of the formulas occurring in the path. The global valued structure for X associated to this path is $(\mathbb{N}^*, S^*, v, id)$ where*

- *$\mathbb{N}^*$ is the set of positive natural numbers*
- *S is the set of edges (simple or double) of the path Γ and S^* is the reflexive and transitive closure of the relation S.*
- *v is a mapping of X in $2^{\mathbb{N}^*}$ defined by*
  If $x \in X$ then $v(x) = \{i \mid$ the hypothesis $i : x$ is on $\Gamma\}$

– *id is the identity mapping on* $\mathbb{N}^*$

Property 5. Let Γ a final and complete path and let S be the set of edges (simple or double) of Γ. If a hypothesis $i : \Box A$ is on Γ, then for all integers j such that $i\ S^* j$, the hypothesis $j : A$ is on Γ.

Proof. First we prove that :
(1) If the hypothesis $i : \Box A$ is on Γ, then for all integers j such that $i\ S^* j$, the hypothesis $j : \Box A$ is on Γ.
Assume that the hypothesis $i : \Box A$ is on Γ and let P(n) be the following property : for all integer j such that $i\ S^n j$, the hypothesis $j : \Box A$ is on Γ.
The property P(0) is trivial. We assume P(n) and we prove P(n+1).
Assume that $i\ S^{n+1} j$. There exists an integer k such that $i\ S^n k$ and $k\ S\ j$. By the induction assumption P(n), we know that the hypothesis $k : \Box A$ is on Γ.
Since Γ is final, this hypothesis is marked.
Since Γ is complete, from the property (c4) of Definition 15 and from the existence of an edge from k to j, we conclude that $j : \Box A$ is on Γ.
Consequently, P(n) is true for all natural numbers n, and this implies the property (1).
Now we can achieve the proof of the property 5.
Assume that the hypothesis $i : \Box A$ is on Γ and that $i\ S^* j$.
From the property (1), we deduce that the hypothesis $j : \Box A$ is on Γ.
Since Γ is final,this hypothesis is marked and since Γ is complete, from the property (c4) of Definition 15, we deduce that the hypothesis $j : A$ is on the path. $\Box$

Definition 17 (Size of a formula). *The size of a formula is the the number of occurrences of connectives and variables of the formula.*

Lemma 2 (Satisfiability of a final and complete path). *A final and commplete path is satisfied by the global valued structure associated to it.*

Proof. Let Γ be a final and complete path and let a be $(\mathbb{N}^*, S^*, v, id)$ the global valued structure associated to it (see Definition 16). By definition of a, it satisfies the edges of the path. It is enough to prove that a satisfies the hypotheses of the path. Since id is the identity function, we have to prove that :
(1) If the hypothesis $i : A$ is on Γ, then $i \in v(A)$.
We prove this property above by induction on the size of A.
Let P(k) be the following property :
If A is a formula of size k and if the hypothesis $i : A$ is on Γ then $i \in v(A)$.
(2) Assume P(k) for every k such that $k < n$ and prove P(n).
(3) Assume that A is a formula of size n and that the hypothesis $i : A$ is on Γ.
We must prove that $i \in v(A)$.
We consider three cases.

– $n = 0$
  Since there is no formula of size 0, the assumption (3) is false, consequently we have $i \in v(A)$.

- $n = 1$
  A is a variable and by definition of v, we have $i \in v(A)$.
- $n > 1$
  We consider all possible forms of A :
  - $A = \neg B$ and B is a variable
    Since the path Γ is final (Definition 9), Γ is not closed and the hypothesis $i : B$ is *not* on Γ, hence by definition of v, $i \notin v(B)$, consequently we have $i \in v(A)$.
  - $A = \neg\neg B$
    Since the path Γ is final, the hypothesis $i : A$ is marked. Since Γ is complete, from the condition (c3) of Definition 15, we know that the hypothesis $i : B$ is on Γ. By induction hypothesis (2), we have $i \in v(B)$. Since v(A) = v(B), we have $i \in v(A)$.
  - $A = \neg(B \wedge C), A = (B \wedge C), A = \neg\Box B$
    All theses cases are similar to the previous one.
  - $A = \Box B$
    Let j be an integer such that $i\ S^* j$. From the property 5, we know that the hypothesis $j : B$ is on Γ. By induction hypothesis (2), we have $i \in v(B)$. Consequently we have $i \in v(A)$.

$\Box$

Definition 18 (Pointed valued structure associated to a path). *Let Γ be a path and X the set of variables of the formulas occurring on the path. We suppose that 1 is a state of Γ. The pointed valued structure associated to this path is the 4-tuple $(D, S^*, v, 1)$ where*

- *D is the set of states occurring on Γ*
- *S is the set of edges (simple or double) of the path Γ and S^* is the reflexive and transitive closure of the relation S.*
- *v is a mapping of X in 2^D defined by*
  If $x \in X$ then $v(x) = \{i \mid$ the hypothesis $i : x$ is on $\Gamma\}$

Theorem 2 (Completeness). *If a final tableau can be constructed from a formula, this formula is satisfied by the pointed valued structure associated to a final path of the tableau.*

Proof. Let A be a formula. Let T be a final tableau constructed from A (Definition 13). Since the rules do not suppress any hypothesis, each path of the tableau T has the hypothesis $1 : A$.
Since T is a final tableau, it has a final path Γ. By the Lemma 2 this path is complete.
Let α be $(\mathbb{N}^*, S^*, v, id)$ the global valued structure associated to Γ. By Lemma 2, α satisfies Γ. Since the hypothesis $1 : A$ is on Γ, we have :
(1) $1 \in v_I(A)$ where I is the interpretation $(\mathbb{N}^*, S^*, v)$ (see Definition 4)
Let α' be $(D, S^*, v, 1)$ the pointed valued structure associated to Γ. Let J be the interpretation (D, S^*, v).
By structural induction on formulas, we prove easily that :

(2) for all formulas B, for all i $\in D$, $i \in v_I(B)$ if and only if $i \in v_J(B)$.
From (1) and (2) and from $1 \in D$, we deduce that $1 \in v_J(A)$. Consequently the pointed valued structure associated with the path Γ satisfies the formula A. $\square$

3.3 Termination proof

We show only the key-point of this proof. The number of states of each path of a tableau constructed from a formula is finite.

Lemma 3 (Subformula lemma). *In a tableau constructed from a formula A, the formulas of the tableau hypotheses are subformulas of A or negations of these subformulas.*

Proof. By induction similar to the previous proofs. $\square$

Remark 3. If we use simplification rules (as suggested by [9]), the subformula lemma is no more verified. Consequently it is no sure that application of simplification rules could give a better algorithm.

Lemma 4 (All sets of formulas associated to states are different). *Let Γ be a path of a tableau constructed from a formula and let n be the largest state of Γ. The set Γ_i of formulas associated to a state i occurring on Γ is indicated by Definition 12. The path Γ verifies the following properties :*

(a) *for i from 1 to n, the sets Γ_i are disjoints one from another*
(b) *for i from 1 to n-1, the set Γ_n is not included in Γ_i*

Proof. First we show that :
(1) If a path verifies the properties (a) and (b), then a predecessor of this path verifies also these properties.

Let Γ be a path whose largest state is n and which verifies the properties (a) and (b).
We consider the two following cases:

- Let Γ' a predecessor of Γ by one the rules, except $\neg\Box$. Only the set of formulas of state n has been increased :
  - n is still the largest state of Γ'
  - for i from 1 to n-1, $\Gamma'_i = \Gamma_i$
  - $\Gamma_n \subseteq \Gamma'_n$

  Let us assume that Γ' does not verify the properties (a) and (b). That would imply that there exists i such that :
  (2) $i < n$ and $\Gamma'_n \subseteq \Gamma'_i$.
  From (2), from $\Gamma'_i = \Gamma_i$ $(1 \leq i \leq n-1)$ and from $\Gamma_n \subseteq \Gamma'_n$, we deduce that $\Gamma_n \subseteq \Gamma_i$, which is contrary to the assumption that Γ verifies the properties (a) and (b).
  Consequently, Γ' verifies the properties (a) and (b).

- Let Γ' a predecessor of Γ by the rule $\neg\Box$. We consider the two following cases :
  - If Γ' is obtained without creation of a new state, Γ' has no new hypothesis and consequently verifies (a) and (b).
  - If the rule has created the state n+1, we have added to Γ' only hypotheses of state n+1 with the *condition* that for i from 1 to n, $\Gamma'_n \not\subseteq \Gamma'_i$. Since for i from 1 to n, $\Gamma'_i = \Gamma_i$, the path Γ' verifies the properties (a) and (b).

Let A be a formula. The path I(A) having the unique hypothesis $1 : A$ verifies trivially the properties (a) and (b). From (1), it follows that every path of a tableau constructed from A verifies the properties (a) and (b). $\Box$

Now we can give a limit to the number of states of the paths of a tableau constructed from a given formula.

Theorem 3 (Termination). *The number of states of a path of a tableau constructed from a formula of size k is at most* 2^{2k}.

Proof. Let A be a formula of size k (see Definition 17). Let E be the set of all subformulas of A and the negations of these subformulas. E has at most $2k$ elements. Let Γ a path of a tableau constructed from A and let n the largest state of Γ.
From Lemma 3, we know that for i from 1 to n, every set Γ_i (see Definition 12) is subset of E. From Lemma 4, we know that all the sets Γ_i are different.
Consequently $n \leq 2^{2k}$ $\Box$

Conclusion

We have proved the correctness of a decision procedure for KT4, based on the progressive construction of a pointed valued structure. This proof shows that the application of a rule which can lead to the creation of a new state, must be allowed if the other rules can not be applied. Let us remind the principles of the proof :

- The coherence of the rules is proved by discarding the *double* edges produced by the unwinding test (see 2)
- The *double* edges are used only in the construction of the final pointed valued structure

We hope to extend easily these principles to other modal logics having transitive operators.

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